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José Manuel Giménez Gómez
Josep E. Peris
Begoña Subiza

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Departament d'Economia
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Universitat Rovira i Virgili
Facultat d'Economia i Empresa
Av. de la Universitat, 1
43204 Reus
Tel.: +34 977 759 811
Fax: +34 977 758 907
Email: sde@urv.cat

CREIP
www.urv.cat/creip
Universitat Rovira i Virgili
Departament d'Economia
Av. de la Universitat, 1
43204 Reus
Tel.: +34 977 758 936
Email: creip@urv.cat

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A 'solidarity' approach to the problem of sharing a network cost

José M. Giménez-Gómez,[‡] Josep E. Peris[†] and Begoña Subiza[†]

[‡] *Universitat Rovira i Virgili, Departament d'Economia and CREIP, Av.Universitat 1, 43204 Reus, Spain. (josemanuel.gimenez@urv.cat)*
[†] *Universitat d'Alacant, Mètodes Quantitatius i Teoria Econòmica, IUDESP, 03080 Alacant, Spain. (peris@ua.es, subiza@ua.es)*

Abstract

A *minimum cost spanning tree* problem analyzes the way to efficiently connect individuals to a source when they are located at different places. Several rules have been defined to solve this problem. Our objective here is to propose a new approach that differentiates some costs that may deserve compensations (*involuntary costs*) from some other connection costs that may be considered *voluntary*. We therefore define a *solidarity egalitarian* solution, through which, the total cost is allocated by considering pay-backs to equalize the involuntary costs, thus fulfilling the weak stability condition of *individual rationality*.

Keywords: Minimum cost spanning tree, Solidarity, Cost sharing, Egalitarian
JEL classification: C71, D63, D71.

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1. Introduction

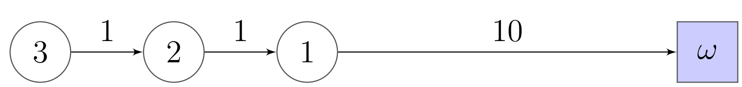
We consider a situation in which some individuals, located at different places, want to be connected to a source in order to obtain a good or a service. Each link connecting any two individuals, or connecting each individual to the source, has a specific fixed cost. Moreover, individuals do not mind being connected to the source, either directly or indirectly through other individuals. This situation is known as the *minimum cost spanning tree problem* (hereafter, the *mcst* problem) and it is used to analyze different actual issues, such as telephone, cable TV or water supply networks.

There are several methods for obtaining a way of connecting agents to the source so that *the total cost of the selected network is minimum* (see, for instance, Prim (1957)). Once the minimum cost network is built, its cost must be allocated among the individuals. There is extensive literature on this issue and several solutions have been proposed. In general, these solutions take the cost of every link in the network into account and, moreover, all of the costs are equally relevant. Contrary to this trend, we consider a model in which some costs in the network deserve different treatment.

In many contexts, either the distribution or the redistribution of some resources or costs consider not only the individuals' inherent characteristics, but also their performed features (referred to as skills or effort in Bossert (1995), or voluntary contributions in Baranski (2016), among others). Following this line, we enrich the classical model of *mcst* problems by adding two functions that determine the characteristics of the individuals: one of these functions determines the costs attributed to the individual's voluntary actions (*voluntary* decisions about the cost) and the other, indicates the costs due to circumstances over which the individual has no responsibility (*involuntary* aspects of the optimal network).

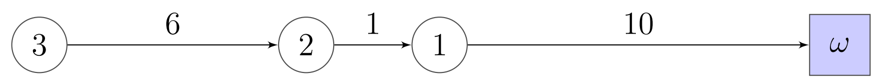
In doing so, we add new variables but, at the same time, we simplify the cost allocation problem since we ignore the cost of most of the unused connections in the optimal network. This is known as the *reductionist* approach, and some solutions to *mcst* problems are defined under this approach. The following examples illustrate our idea.

[1] *Consider a set of three houses in a row (at the same distance each one from the other). A new water supply ω is now located at one end of the row. The cost of each link among the houses is* 1 *monetary unit and the nearest house to the supply may connect directly with a cost of* 10 *units; the second house has a direct cost of* 11 *units; and the cost of directly connecting the farthest house is* 12 *units. The total (minimum) cost of connecting the three houses to the water supply is* 12 *units.*



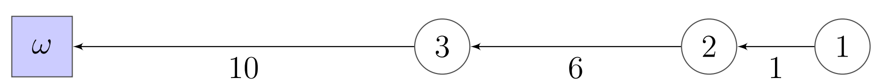
In this situation, as individuals have no responsibility regarding the location of the water supply, so those who are further from the source *should be compensated. A reasonable sharing of this cost is egalitarian: 4 units to each individual. Many solutions, defined in the minimum cost spanning tree literature, propose a non egalitarian sharing of the cost in this problem.*

[**2**] *Consider a similar situation, but now individual 3 has voluntarily decided to live in a luxury mansion on the outskirts of the town, so now the new situation is as depicted in the following graph:*



As in the previous case, this individual should be compensated because the water supply is at the (other) extreme of the row. But her voluntary decision to live away from other people (which has increased the total cost of the network by 5 units) does not deserve compensation. Hence, an equal sharing of the total cost $C_m = 17$ does not seem as natural now as it did before. A reasonable proposal would be to assign the extra cost to individual 3 (due to her voluntary decision), and to share the remainder equally among them; that is, the cost allocation $(4, 4, 9)$.

[**3**] *An interesting question is to compare the dual situation of this problem, in which the source is located near individual 3. Under a fair system of sharing the total cost, this new situation should be considered equivalent to the previous one, and the cost allocation should be the same in both problems.*



Our paper considers that each individual has a *voluntary characteristic* (represented by a real number) that could be determined, for instance, by their connection costs through other individuals: they decide if they want to live close to other people, or away from them. On the other hand, there is an *involuntary characteristic* that does not depend on their decisions (for instance, it could be represented by the cost of connecting the individual to the source: when they do not decide where the source is located, nor the network implementation).[1]

Our main criterion in allocating the total cost of the optimal network is that of *egalitarianism*: whenever possible, all individuals should pay the same amount, as long as they do not cause any unnecessary additional costs. That is, only involuntary costs should be equally shared, and costs due to voluntary decisions should affect only the individuals concerned. On the other hand, no individual can be allocated a cost that is greater than their own direct connection to the source, that is, the *stand alone* condition (usually known as individual rationality), because, in such a case, the individual would be better off acting on their own.

Accordingly, we define a *Solidarity Egalitarian* rule that tries to compensate individuals for extra costs that they are not responsible for. Apart from analyzing its properties (axiomatic analysis), we show how our proposal performs in the situation in previous examples and obtain that it provides an equal sharing in the symmetric situation (case [1]) and, not surprisingly, provides the same allocation of the cost $C_m = 17$ in both non symmetric problems (cases [2] and [3]).

The rest of the paper is organized as follows. Section 2 presents the formal *mcst* problem. Section 3 presents our approach and our solution concept is introduced. Its properties are analyzed in Section 4. Section 5 shows a particular case. Some additional comments and possible extensions appear in Section 6. An appendix presents some notions on fair redistribution that are used in our previous discussion.

2. Preliminaries: Minimum cost spanning tree problem

A *minimum cost spanning tree* problem involves a finite set of *agents*, $N = \{1, 2, \ldots, n\}$, who need to be connected to a *source* ω. We denote by N_ω the set of agents and the source, i.e. $N \cup \{\omega\}$. The agents are connected by edges and for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the edge e_{ij} connecting

[1] In a recent paper by Giménez-Gómez et al. (2014) the minimum connection cost to each individual (the cost of connecting each individual with their *nearest* neighbor) and the cost of connecting the individual directly to the source have been considered to make a link between the literature on *mcst* problems and that on *conflicting claims* problems.

agents $i, j \in N$. Following the notation in Kar (2002), c_{ii} represents the cost of directly connecting agent i to the source, for all $i \in N$ (the *stand alone* cost). We denote by $\mathbf{C} = [c_{ij}]_{n\times n}$ the $n \times n$ symmetric cost matrix. The *mcst* problem is represented by the pair $(N_\omega, \mathbf{C})$. We denote by $\mathcal{N}_n$ the set of all *mcst* problems with n individuals.

A *spanning tree* over $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ is an undirected graph p with no cycles, which connects all elements of N_ω. We can identify a spanning tree with a function $p : N \to N_\omega$ so that $p(i)$ is the agent (or the source) to whom i connects, and defines the edges $e^p_{ij} = (i, p(i))$. In a spanning tree each agent is (directly or indirectly) connected to the source ω; that is, for all $i \in N$ there is some $t \in \mathbb{N}$ such that $(p \circ\ ..^{(t}. \circ p)(i) = p^t(i) = \omega$. Moreover, given a spanning tree p, there is a single path from any i to the source for all $i \in N$, given by the edges $(i, p(i)), (p(i), p^2(i)), \ldots, (p^{t-1}(i), p^t(i) = \omega)$. The cost of building the spanning tree p is the total cost of the edges in this tree; that is,

$$C_p = \sum_{i=1}^{n} c_{ip(i)}.$$

Prim (1957) provides an algorithm that solves the problem of connecting all the agents to the source such that *the total cost of the network is minimal*.[2] The solution achieved, the *minimum cost spanning tree*, may not be unique. Denote by m a tree with the minimum cost and by C_m its cost. That is, for all spanning tree p,

$$C_m = \sum_{i=1}^{n} c_{im(i)} \leqslant C_p = \sum_{i=1}^{n} c_{ip(i)}.$$

Once the *minimum cost spanning tree* is constructed, an important issue is how to allocate the associated cost C_m among the agents.[3]

[2] This algorithm has n steps. First, we select the agent i with the lowest connection cost to the source. In the second step, we select an agent in $N \backslash \{i\}$ with the smallest cost, either directly to the source or to agent i, who is already connected. We continue thus until all agents are connected, i.e., at each step, connecting an agent who is still not connected to one who already is, or directly to the source.

[3] Actual situations reveal that agents do not necessarily agree on how to distribute this cost, in which case the social optimum is not implemented, so a more expensive cost than necessary is incurred to build a tree that connects the agents to the source (for an example, see Bergantiños and Lorenzo (2004); see Hernández et al. (2016) for a discussion about individual and social optimality).

A *sharing rule*, also called a *solution* for the cost allocation problem, is a function that proposes for any *mcst* problem $(N_\omega, \mathbf{C})$ an allocation $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$, such that[4]

$$\sum_{i=1}^{n} \alpha_i = C_m.$$

It is noteworthy that many solutions have been defined in the *mcst* literature: for instance *Bird* (Bird, 1976), *Serial* (Moulin and Shenker, 1992), *Kar* (Kar, 2002) or *Folk* (Bergantiños and Vidal-Puga, 2007).

Remark 1. *Some of these solutions take all the possible connections to the network into account, although most of these connections are not used in the optimal tree. Nevertheless, other solutions, such as the Bird or the Folk solutions, are obtained by only considering some of the connection costs. Specifically, the Bird solution only considers the cost of the link each individual uses in the optimal network while the cost of the other edges are ignored; an egalitarian sharing only considers the total cost of the network; this is also the case of the Serial solution (Moulin and Shenker, 1992) which only considers the connections being used by the agents. This is called a reductionist approach, which ignores some of the available information and reduces the parameters of the problem (and the complexity in the computation).*

In this context, demanding that the maximum amount that should be allocated to any individual cannot exceed the cost of connecting directly to the source (individual rationality, or stand alone stability) is a compulsory requirement, since, if some individual is allocated a cost that is greater than their direct cost to the source, they are better off acting by themselves and not cooperating in building the optimal network.

Axiom 1. *Individual Rationality **(IR)**: Given a mcst* $(N_\omega, \mathbf{C})$, *an allocation* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\sum_{i=1}^{n} \alpha_i = C_m(N_\omega, \mathbf{C})$, *is said to be individually rational if, for all* $i \in N$,

$$\alpha_i \leqslant c_{ii}.$$

[4] In some contexts the non-negativity condition is not required. We do not follow this approach. This question is related to the assumption of *property* or *non-property rights* on the locations that individuals occupy (see, for instance, Bogomolnaia and Moulin (2010)).

Note that an equal sharing of the total cost may imply that some of the agents can be charged a cost that is greater than their direct cost to the source, c_{ii}, and, in this case, dividing the network cost equally may fail to fulfill individual rationality.

Once the minimum spanning tree m is built with a cost C_m, we may consider that, initially, each individual pays for the edge they use to be connected, $(i, m(i))$, so they are charged the cost $c_{im(i)}$. This allocation is always individually rational, from the way in which the minimum cost spanning tree m is obtained.

From this initial distribution of the minimum cost, any cost allocation α is merely a *redistribution* of the total cost, since $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} c_{im(i)} = C_m$. Therefore, it is possible to express any solution for the cost allocation problem in the form:

$$\alpha_i = c_{im(i)} + x_i, \qquad x_i \in \mathbb{R}, \quad \sum_{i=1}^{n} x_i = 0, \tag{1}$$

where x_i can be interpreted as a *tax* or a *subsidy*, depending on the sign it has (positive or negative, respectively), the individual pays/receives from their initial contribution.

3. The setting: a solidarity approach in *mcst* problems

Let us consider a finite population $N = \{1, 2, \ldots, n\}$, that wants to be connected to a source ω. Let $\mathbf{C}$ be the connection cost matrix, and let m be the minimum cost spanning tree associated with this cost matrix, and C_m the minimum cost. In any *mcst* problem $(N_\omega, \mathbf{C})$, we assume that each agent $i \in N$ is identified by two characteristics (r_i, s_i) representing, respectively, the *voluntary* and *involuntary* decision of individual i. These characteristics are defined by means of two real valued functions that depend on the available information about individual i, namely the vector of connection costs of this individual $\hat{c}_i = (c_{ij}, j \in N)$. These functions then take the general form

$$r, s : \mathbb{R}_+^n \to \mathbb{R}_+ \quad r_i = r(\hat{c}_i),\ s_i = s(\hat{c}_i), \quad i = 1, 2, \ldots, n.$$

As mentioned in Section 1 the voluntary characteristic represents some decision that affects the total cost of the network and depends only on the individual's decision: either to live near or far from other people; to be located near the source (if the source has been set up before the individual selected their location); etc. On the other hand, some characteristics are outside the control of the individual, such as the structure of the optimal network, or the location of the source when individuals are previously settled, etc.

Thus, we extend the *mcst* problem by adding two new components: the functions representing voluntary and involuntary characteristics, so that the problem is $(N_\omega, \mathbf{C}; r, s)$. Moreover, as aforementioned, initially we consider that a cost is allocated to each individual that does not depend on their characteristics but rather, on the implemented network; that is a function that assigns to each individual the cost they incur in the *mcst* to be connected to the network, namely $f(r_i, s_i) = c_{im(i)}$.

We are interested in obtaining a (solidary) fair redistribution mechanism that equalizes the costs incurred due to unequal and involuntary characteristics, but we do not compensate voluntary differences. In this sense, note that any alternative allocation of the total cost takes the form shown in Equation (1), so it is a redistribution of the cost. In so doing, we use some results obtained in fair compensation models (see Fleurbaey and Maniquet (2011) for a survey on this literature).[5]

The (egalitarian) mechanism we propose is based on the following two natural properties that determine how it works with respect to the function r that establishes the voluntary characteristics.

[**ED**] A sharing rule α satisfies **equal distribution** if for any *mcst* problem $(N_\omega, \mathbf{C}; r, s)$,

$$r_i = r_j \quad \text{for all } i, j \in N \quad \Rightarrow \quad \alpha_i = \alpha_j \quad \text{for all } i, j \in N.$$

This property implies that if all of the individuals are *identical* with respect to the *voluntary* characteristic, in which case, each of them should be charged an equal part of the total cost of the network

$$\alpha_i = \frac{1}{n} C_m.$$

[**IM**] A sharing rule α satisfies **individual monotonicity** if given two *mcst* problems that differ in some costs, $(N_\omega, \mathbf{C}; r, s)$ and $(N_\omega, \mathbf{C}'; r', s')$, such that $r_i = r'_i$ for all $i \in N$, $i \neq k$, and $r_k \neq r'_k$ for some $k \in N$, then

$$\alpha'_i = \alpha_i \quad \text{for all } i \in N,\ i \neq k.$$

[5] The main definitions and results we use regarding redistribution and fair compensation appear in the Appendix.

This property requires that the extra cost due to some individual's modification of their *voluntary* characteristic, does not affect the other individuals.

The following proposition, which is a direct consequence of a general result in Bossert (1995) (see Proposition 6 in the Appendix), establishes the form of a sharing rule in *mcst* problems that satisfy these axioms.

Proposition 1. *Let α be a sharing rule in mcst problems satisfying (**ED**) and (**IM**). Then, function $f(r_i, s_i)$ is additively separable in voluntary and involuntary characteristics, $f(r_i, s_i) = g(r_i) + h(s_i)$, and*

$$\alpha_i = g(r_i) + \frac{1}{n}\left(\sum_{i=1}^{n} h(s_i)\right) \qquad i = 1, 2, \ldots, n. \tag{2}$$

Note that,

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} c_{im(i)} = C_m \Rightarrow \sum_{i=1}^{n} h(s_i) = C_m - \sum_{i=1}^{n} g(r_i)$$

so that the exact form of the function $h(\cdot)$ does not have to be known, and then the redistribution mechanism takes the following form:

$$\alpha_i = g(r_i) + \frac{1}{n}\left(C_m - \sum_{i=1}^{n} g(r_i)\right) \qquad i = 1, 2, \ldots, n. \tag{3}$$

Therefore, only the voluntary characteristics play a relevant role in the cost distribution. This proposal assigns the cost due to each individual as a result of their voluntary characteristic, and then the extra cost (or benefit) is equally shared among all of the agents. The following definition presents formally this mechanism.

Definition 1. *For any mcst problem $(N_\omega, \mathbf{C}; r, s)$, the Egalitarian redistribution rule is defined by*

$$\alpha_i^{ER}(N_\omega, \mathbf{C}; r, s) = g(r_i) + \frac{1}{n}\left(C_m - \sum_{i=1}^{n} g(r_i)\right) \qquad i = 1, 2, \ldots, n.$$

Although this mechanism fulfills the axioms on which is based, ***(ED)*** and ***(IM)***, it may fail to be individually rational, an essential property in *mcst* problems (see example in Section 5.3). This is why we propose a modification of the Egalitarian redistribution mechanism α^{ER}, which we denote by π, in order to obtain individual rationality; that is, that no individual should pay more than the cost of their direct connection to the source, $\alpha_i \leqslant c_{ii}$.

Definition 2. *For any mcst problem* $(N_\omega, \mathbf{C}; r, s)$*, the Solidarity Egalitarian rule* π *is defined in several steps:*

Step 1: Compute the Egalitarian redistribution solution:

$$\phi_i(1) = \alpha_i^{ER}(N_\omega, \mathbf{C}; r, s) = g(r_i) + \frac{1}{n}\left(C_m - \sum_{i=1}^{n} g(r_i)\right) \quad i = 1, 2, \ldots, n.$$

Step 2: Check Individual Rationality of $\phi(1)$

case a) If $\phi_i(1) \leqslant c_{ii} \quad \forall i \in N$*, define* $\pi(N_\omega, \mathbf{C}; r, s) = \phi(1)$.

case b) In another case, let $N_1 = \{k \in N : \phi_k(1) < c_{kk}\}$*, and define*

$$\phi_i(2) = \begin{cases} c_{ii} & \forall\, i \notin N_1 \\ \phi_i(1) + \dfrac{1}{n_1} \displaystyle\sum_{j \notin N_1} (\phi_j(1) - c_{jj}) & \forall\, i \in N_1 \end{cases}$$

where n_1 *is the number of individuals in* N_1.

Step 3: Check Individual Rationality of $\phi(2)$

case a) If $\phi_i(2) \leqslant c_{ii} \quad \forall i \in N$*, define* $\pi(N_\omega, \mathbf{C}; r, s) = \phi(2)$.

case b) In another case, let $N_2 = \{k \in N_1 : \phi_k(2) < c_{kk}\}$*, and define*

$$\phi_i(3) = \begin{cases} c_{ii} & \forall\, i \notin N_2 \\ \phi_i(2) + \dfrac{1}{n_2} \displaystyle\sum_{j \notin N_2} (\phi_j(2) - c_{jj}) & \forall\, i \in N_2 \end{cases}$$

where n_2 *is the number of individuals in* N_2.

Step 4: Continue with the above process until $\phi_i(t) \leqslant c_{ii}$ *for all* $i \in N$*, and then define* $\pi(N_\omega, \mathbf{C}; r, s) = \phi(t)$.

From the above definition, as the next result shows, it is straightforward to see that the redistribution mechanism π may be obtained throughout the well know *constrained equal awards rule*, used in conflicting claims problems.[6] The following result shows this relation.

Proposition 2. *For all mcst problem* $(N_\omega, \mathbf{C}; r, s)$*, and all* $i \in N$*,*

$$\pi_i(N_\omega, \mathbf{C}; r, s) - g(r_i) = CEA_i(M, d) \tag{4}$$

where

$$M = C_m - \sum_{i=1}^{n} g(r_i), \quad d_i = c_{ii} - g(r_i).$$

4. Axiomatic analysis

Throughout this section we will consider that the function that defines the voluntary characteristic is non-decreasing in the cost for each individual. That is, if the costs for individual i are $\hat{c}_i = (c_{ij},\ j \in N)$, and $r_i = r(\hat{c}_i)$, then the following condition is fulfilled:

$$\hat{c}_i \leqslant \hat{c}_j \quad \Rightarrow \quad r_i \leqslant r_j \qquad \forall i, j \in N.$$

We also assume that function g is non-decreasing: if the voluntary characteristic of some individual is greater than that of another, then the voluntary part allocated must also be greater:

$$r_i \leqslant r_j \quad \Rightarrow \quad g(r_i) \leqslant g(r_j) \qquad \forall i, j \in N.$$

Under these assumptions, we analyze the Solidarity Egalitarian rule as defined by Equation (4).

[6] The *constrained equal awards rule*, CEA, in claims problems in which some endowment M is distributed among several individuals who have some *claim* d_i on it, such that $\sum_{i \in N} d_i > M$, is expressed in the following way:

$$CEA_i(M, d) = \min\{d_i, \lambda\} \text{ where } \lambda \text{ is chosen such that } \sum_{i \in N} CEA_i(M, d) = M.$$

In Giménez-Gómez and Peris (2015) a redistributive mechanism is obtained by using conflicting claims rules.

When observing sharing rules for *mcst* problems, we realize that several requirements are recognized in the literature as being compelling.[7] It is clear that the Solidarity Egalitarian rule fulfills individual rationality. Apart from this axiom, one of the properties we are interested in is the so-called *Ranking*, which can be stated as follows: *an agent who is more expensive to connect than another one must not pay less.* As mentioned in Bogomolnaia and Moulin (2010) "all reasonable solutions discussed in the literature satisfy *Ranking*." Formally, this property is defined in the following way:

Axiom 2. *Ranking **(RKG)**. A solution satisfies **RKG** if for any mcst problem* $(N_\omega, \mathbf{C})$ *such that* $c_{ik} \leqslant c_{jk}$ *for all* $k \in N$*, proposes an allocation* $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$ *such that* $\alpha_i \leqslant \alpha_j$*.*

Ranking implies the weak property of *Equal Treatment of Equals.*

Axiom 3. *Equal Treatment of Equals **(ETE)**. A solution satisfies **ETE** if for any mcst problem* $(N_\omega, \mathbf{C})$ *such that* $c_{ik} = c_{jk}$ *for all* $k \in N$*, proposes an allocation* $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$ *such that* $\alpha_i = \alpha_j$*.*

Proposition 3. π *satisfies **RKG** and **ETE**.*

Proof. Note that, for each $i \in N$, $\pi_i = g(r_i) + CEA_i\,(M, d)$, where M and d are defined as in Proposition 2. Moreover, from the assumptions, we know that $g(r_i) \leqslant g(r_j)$. We distinguish two cases:

1. If $d_i \leqslant d_j$, that is $c_{ii} - g(r_i) \leqslant c_{jj} - g(r_j)$, since CEA satisfies *Order preservation* (Aumann and Maschler (1985); see Thomson, 2015, for formal definitions),

$$CEA_i(M, d) \leqslant CEA_j(M, d), \text{ and } \pi_i \leqslant \pi_j.$$

2. If $d_i > d_j$, then
   (a) If $CEA_j(M, d) = d_j = c_{jj} - g(r_j)$, then $\pi_j = c_{jj} \geqslant c_{ii} \geqslant \pi_i$.
   (b) If $CEA_j(M, d) < d_j$, then $CEA_i(M, d) = CEA_j(M, d)$ so $\pi_i \leqslant \pi_j$.

As ***RKG*** implies ***ETE***, then π also fulfills this property. ■

The following property (Bogomolnaia and Moulin, 2010) expresses how the cost allocated to an individual should change: "*cost shares should be weakly responsive to one's own connecting costs.*"

[7] See Bergantiños and Vidal-Puga (2007, 2008) and Bogomolnaia and Moulin (2010).

Axiom 4. *Cost Monotonicity **(CM)**. A solution satisfies **CM** if for any two mcst problems* $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$, *such that for all* $i \in N$, $j, k \in N_\omega$

$$c_{ij} < c'_{ij} \text{ and } c_{lk} = c'_{lk} \ \forall\, (l, k) \neq (i, j)$$

proposes allocations $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$ *and* $(\alpha'_1, \alpha'_2, \ldots, \alpha'_n) \in \mathbb{R}^n_+$, *such that for all* $i \in N$, $\alpha_i \leqslant \alpha'_i$.

Proposition 4. π *satisfies **CM**.*

Proof. For each $i \in N$, $\pi_i = r(r_i) + CEA_i(M, d)$, where M and d are defined as in Proposition 2. As CEA satisfies *Order preservation* and *Claims monotonicity* (see Thomson, 2015), if $c_{ij} < c'_{ij}$ it is straightforward to obtain that $\pi_i \leqslant \pi'_i$. So ***CM*** is satisfied. ■

The following property requires that if any individual is connected to the source through their own voluntary cost, then no redistribution is made and each one pays their own voluntary connection cost.

Axiom 5. *Minimum allocation property **(MA)**. A solution satisfies **MA** if for any mcst problem* $(N_\omega, \mathbf{C}; r, s)$ *such that* $\sum_{i=1}^n g(r_i) = C_m$, *then,*

$$\alpha_i(N_\omega, \mathbf{C}; r, s) = g(r_i), \quad \text{for all } i \in N.$$

Proposition 5. π *satisfies **MA**.*

Proof. Note that the assumption $\sum_{i=1}^n g(r_i) = C_m$ implies $M = 0$. From Proposition 2 we then obtain $\pi_i = g(r_i) + 0 = g(r_i)$, for all $i \in N$, so the Solidarity Egalitarian rule fulfills this property. ■

5. Voluntary characteristics: some particular cases

In this section, we analyze a particular case of the Solidarity Egalitarian rule by fixing a specific value to function $g(\cdot)$, and proposing a specification for function $r(\cdot)$, that defines the voluntary characteristics. This fact will allow us to obtain the allocation provided by our solution in some examples, and to compare it with other existing solutions in the literature. It should be noticed again that function $h(\cdot)$ in Proposition 1 can be eliminated and, therefore, the involuntary characteristics do no play any relevant role in the discussion.

5.1. ***Function*** $g(\cdot)$

To simplify the model, we consider the function $g(x) = x$. This implies that each individual pays the part that the voluntary characteristic assigns to them. The remainder (positive or negative) is equally shared according to the constrained equal awards rule. That is,

$$\pi_i(N_\omega, \mathbf{C}; r, s) = r_i + CEA_i(M, d) \tag{5}$$

where

$$M = C_m - \sum_{i=1}^{n} r_i, \quad d_i = c_{ii} - r_i.$$

5.2. ***Function*** $r(\cdot)$

In order to obtain a specific expression in Equation (5) we need to define the voluntary characteristic r_i of each individual. This value should represent a decision that *does not deserve compensation*. There exist different possibilities for this component and, depending on the context, one of these possibilities could be more appropriate than the others. We illustrate some of them.

1. As mentioned in Section 1, agents may choose where to live: around other people, or away from them. So, they are responsible for the costs c_{ij}, $i \neq j$. Then, a possibility consists of considering the minimum connection cost of this individual as the voluntary characteristic, that is, the cost of connecting through the nearest neighbor,

   $$r_i = c_{i*} = \min_{j \in N} \{c_{ij}\}.$$

   In this case, the Solidarity Egalitarian rule is defined as:

   $$\pi_i(N_\omega, \mathbf{C}; r, s) = c_{i*} + CEA_i(M, d) \tag{6}$$

   where

   $$M = C_m - \sum_{i=1}^{n} c_{i*}, \quad d_i = c_{ii} - c_{i*}.$$

   We will use this intuitive formulation to discuss the examples.[8]

[8] Although it can be considered a reasonable choice for the voluntary characteristic, this choice may be *unfair* in some circumstances: imagine two individuals who choose to live on the outskirts, but very near each other. Then, their voluntary cost is smaller, harming other individuals who live together in the center of the town.

2. Another possibility consists of considering the average of the connection costs of all the individuals, with or without considering the cost of directly connecting to the source. Then, we have two possibilities:

$$r_i = \frac{1}{n}\left(\sum_{k \neq i} c_{ik}\right) \quad \text{or} \quad r_i = \frac{1}{n}\left(\sum_{k=1}^{n} c_{ik}\right).$$

   In both cases, however, people who live away from the others have a greater voluntary cost.

3. We can also consider that the voluntary characteristic is given by the direct cost to the source, $r_i = c_{ii}$. This case has sense if the source is set prior to the individuals' decisions about where they would prefer to live.

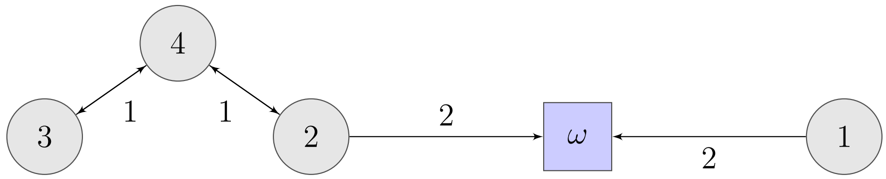
4. It may be that an external authority (possibly, the same one that decides where the source is to be located) decides on a "central point" (for instance, the center of gravity used in location planning) and r_i is the cost of connecting individual i to such a point. As before, this choice makes sense if this central point is determined prior to the individuals' decisions about they would prefer to live.

5.3. ***Example***

We now present an example in which we apply our solidarity egalitarian rule and compare the result it provides with some usual solutions in *mcst* problems.

Example 1. *Typical kinds of mcst problems are the so-called* $2-$*mcst problems in which there are only two possible costs (usually called the low and the high cost). Suppose that individuals can choose to have either low or high-cost connections. In this kind of simple problems each individual has two possible connection costs, and the set* N *of individuals can be partitioned into two sub-sets,* $N = N_1 \cup N_2$*, such that* $N_1 = \{i \in N : c_{ij} = high \text{ for all } j \in N\}$*,* $N_2 = N \backslash N_1$*, then a natural voluntary characteristic is obtained by setting* $r_i = high$ *for* $i \in N_1$*, and* $r_i = low$*, otherwise. This coincides with the minimum connection cost of each individual. Next we provide a numerical example.*

Let us consider the mcst problem defined by (arcs not depicted have a cost $c_{ij} = 2$*):*



A minimum cost spanning tree is given by function m defined as:

$$m(1) = \omega \quad m(2) = \omega \quad m(3) = 4; \quad m(4) = 2; \qquad C_m = 6.$$

Table 1 presents the result of applying some usual sharing rules for mcst problems.

	α_1	α_2	α_3	α_4
Bird	2	2	1	1
Serial	2	4/6	7/6	13/6
Kar	2	3/2	3/2	1
Folk	2	4/3	4/3	4/3

Table 1: Proposals given by *mcst* solutions in Example 1.

If we consider $r_i = c_{i*}$, the minimum connection cost of individual i, the vector of the minimum costs is $r = (2, 1, 1, 1)$, so

$$\alpha^{ER}(N_\omega, \mathbf{C}; r, s) = r + \frac{1}{4}\left(C_m - \sum_{i=1}^{4} c_i^m\right) =$$

$$= (2, 1, 1, 1) + \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(\frac{9}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right),$$

which implies $\alpha_1^{ER} > c_{11}$. That is, the egalitarian sharing rule defined by redistribution α^{ER} does no fulfill individual rationality (Axiom 1). If we now compute the Solidarity Egalitarian rule, we obtain

$$\pi = \left(2, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right).$$

In this case our solution coincides with the Folk solution, but this is not generally the case. Note that the individual with a high cost, $N_1 = \{1\}$, pays the entire connection cost, whereas other individuals are compensated and share the remaining cost equally.

6. Final comments

We have proposed a way of sharing the cost of a network that considers solidarity aspects and treats agents in an egalitarian way, with respect to circumstances for which they are not responsible. Moreover, we have shown that this solution fulfills some appealing properties.

If we go back to the example in Section 1, considering $r_i = \min_{j\in N}\{c_{ij}\}$, the minimum connection cost of individual i, then whenever the situation is symmetric (case [1]) the Solidarity Egalitarian rule provides the sharing $\pi = (4, 4, 4)$, a natural allocation in this situation. Observe in Table 2 that usual solutions in *mcst* problems do not coincide with this proposal.

	α_1	α_2	α_3
π	4	4	4
Bird	10	1	1
Serial	20/6	23/6	29/6
Kar	20/6	23/6	29/6
Folk	4	4	4

Table 2: Proposals given by *mcst* solutions: case [1].

If we analyze the non-symmetric problem we had two different situations, depending on the location of the source (cases [2] and [3]). The first one locates the source near individual 1 and the second near individual 3. In both situations, the Solidarity Egalitarian rule provides the allocation $\pi = (4, 4, 9)$. So the proposed rule satisfies the idea of compensating individuals for the circumstance for which they are not responsible (i.e., the location of the source). Tables 3 and 4 present the result of applying usual *mcst* rules. Note that, in these examples, the *Folk* solution differs from our proposal.

	α_1	α_2	α_3
π	4	4	9
Bird	10	1	6
Serial	20/6	23/6	59/6
Kar	20/6	23/6	59/6
Folk	29/6	29/6	44/6

Table 3: Proposals given by *mcst* solutions: case [2].

	α_1	α_2	α_3
π	4	4	9
Bird	1	6	10
Serial	44/6	38/6	20/6
Kar	44/6	38/6	20/6
Folk	29/6	29/6	44/6

Table 4: Proposals given by *mcst* solutions: case [3].

Finally, if we consider that redistribution is obtained throughout the general case of two real functions $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(r_i, s_i) = g(r_i) + h(s_i).$$

then, under Axioms ***(ED)*** and ***(IM)***, the form of the redistribution mechanism is :

$$\alpha_i = g(r_i) + \frac{1}{n}\left\{\sum_{i=1}^{n} h(s_i)\right\} = g(r_i) + \frac{1}{n}\left\{C_m - \sum_{i=1}^{n} g(r_i)\right\}.$$

In this more general case, agents may pay less than their minimal cost edge, or their allocation may even be negative, for suitable selections of the voluntary characteristics function $g(\cdot)$. As before, this allocation may fail to be individually rational, but a construction like the one we have proposed in defining π allows us to obtain a mechanism that fulfills this property.

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Appendix. Solidarity in redistribution: fair compensation

This Appendix presents the necessary notions on fair redistribution problems previously used. Many redistributive issues involve situations in which initial characteristics make individuals unequal. In general, some of these characteristics call for compensating transfers, and some do not. The literature on redistributive mechanisms and fair compensation tries to obtain allocations that should neutralize the characteristics that elicit compensation and, at the same time, should remain neutral with respect to inequality arising from the influence of characteristics that do not elicit compensation.

We follow the model developed in Bossert (1995). The main idea is that individuals are identified by some characteristics, $y = (y_R, y_S)$, such that features y_R are considered voluntary (and do not deserve compensation), whereas characteristics y_S are involuntary (and deserve compensation). The vector $\boldsymbol{y} = (y^1, y^2, \ldots, y^n)$ is the characteristics profile of the set of agents,

$y^i = (y^i_R, y^i_S)$, $i \in N$. An income function f assigns a positive pre-tax income $f(y^i) \in \mathbb{R}$, which is determined by the characteristics of individual i. Then, the redistribution problem is identified with the pair $\mathcal{E} = (\boldsymbol{y}, f)$.

The purpose of fair compensation models is to find a way of redistributing the pre-tax incomes on the basis of the individuals' characteristics. As mentioned in Bossert (1995), the basic idea is that the effects of involuntary characteristics should be eliminated, whereas the contributions of voluntary characteristics to individual incomes should be preserved. Then, an income redistribution is defined by a mechanism F such that

$$F_i(\mathcal{E}) = f(y^i) + x_i, \qquad x_i \in \mathbb{R}, \quad \sum_{i=1}^{n} x_i = 0.$$

Note that this formulation is equivalent to the one presented in Equation (1). Bossert characterizes the redistribution mechanisms that satisfy some natural axioms.

Axiom 6. ***Equal Distribution***: *A redistribution mechanism F fulfills equal distribution **(ED)** if for any redistribution problem $\mathcal{E} = (\boldsymbol{y}, f)$,*

$$y^i_R = y^j_R \quad \text{for all } i, j \in N \quad \Rightarrow \quad F_i = F_j \quad \text{for all } i, j \in N.$$

Axiom 7. ***Individual Monotonicity***: *A redistribution mechanism F fulfills individual monotonicity **(IM)** if for any two redistribution problems $\mathcal{E} = (\boldsymbol{y}, f)$, $\mathcal{E}' = (\boldsymbol{y}', f)$, such that $\boldsymbol{y}$ and $\boldsymbol{y}'$ coincide except in $y^k_R \neq y'^k_R$ for some $k \in N$, then*

$$F_i(\mathcal{E}') = F_i(\mathcal{E}) \quad \text{for all } i \in N, \ i \neq k.$$

The following result shows that, under these conditions, the redistribution mechanism is completely characterized.

Proposition 6. *(Bossert, 1995) Let F a redistribution mechanism satisfying **(ED)** and **(IM)**. Then, for all redistribution problem $\mathcal{E} = (\boldsymbol{y}, f)$, the income function is additively separable, $f(y_R, y_S) = g(y_R) + h(y_S)$ and*

$$F_i(\mathcal{E}) = g\left(y^i_R\right) + \frac{1}{n}\left(\sum_{i=1}^{n} h\left(y^i_S\right)\right) \qquad i = 1, 2, \ldots, n.$$

The above expression establishes the form of the redistribution mechanism in which each individual's allocation depends on their voluntary characteristics and the rest is shared equally among all of the agents.