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Pedro Calleja
Francesc Llerena

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Departament d'Economia
www.fcee.urv.es/departaments/economia/public_html/index.html
Universitat Rovira i Virgili
Facultat d'Economia i Empresa
Av. de la Universitat, 1
43204 Reus
Tel.: +34 977 759 811
Fax: +34 977 758 907
Email: sde@urv.cat

CREIP
www.urv.cat/creip
Universitat Rovira i Virgili
Departament d'Economia
Av. de la Universitat, 1
43204 Reus
Tel.: +34 977 758 936
Email: creip@urv.cat

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Consistency distinguishes the (weighted) Shapley value, the (weighted) surplus division value and the prenucleolus

Pedro Calleja and Francesc Llerena *

Abstract

On the domain of cooperative games with transferable utility, we investigate how the main results in Hart and Mas-Colell (1989) vary when we replace self consistency by projected consistency or max consistency. As a consequence, we obtain several axiomatic comparison among the (weighted) Shapley value, the (weighted) surplus division solution and the prenucleolus.

1 Introduction

A cooperative game with transferable utility (hereafter game) describes a situation in which a society or community can profit from joint efforts. It consists of a finite set of players and a real-valued function defined on the set of coalitions of players. Assuming that the grand coalition will form, the question is how to allocate the gains from cooperation among the players. A *single-valued solution* (or *rule*) is a mapping that assigns to each game a *feasible* payoff vector, being one of the objectives of the axiomatic method to identify a solution by a set of appealing properties.

Probably, the most relevant single-valued solution is the *Shapley value* (Shapley, 1953b), Sh, which consider that players should be paid according only to their marginal contributions to all coalitions. In front of this marginality principle, the *equal surplus division* solution (Driessen and Funaky, 1991), ES,[1] relies on egalitarian considerations: it assigns to every player what they can achieve for themselves alone, and distributes equally what is left of the gains of cooperation. Both, Sh and ES, satisfy *equal treatment of equals*. This property states that if two players have equal contributions to all coalitions, they must receive the same

*Dep. de Matemàtica Econòmica, Financera i Actuarial, Universitat de Barcelona
Dep. de Gestió d'Empreses, Universitat Rovira i Virgili-CREIP,
e-mail: calleja@ub.edu (Pedro Calleja), francesc.llerena@urv.cat (Francesc Llerena).

[1]The ES solution is also known as the center-of-gravity of the imputation set.

payoff. Nevertheless, in many applications, and because of external features of the players, the assumption that every player has the same abilities may not be appropriated. The *weighted Shapley value* (Shapley, 1953a), Sh^w, and the *weighted surplus division solution* (Calleja and Llerena, 2016), ES^w, take care of this aspect by assigning exogenously each player to a strictly positive weight, representing such abilities.[2] A different prominent rule is the prenucleolus (Schmeidler, 1969), ν_*, that takes specially care of minimizing complains of coalitions to a particular allocation. Interestingly, although the definitions of the (weighted) Shapley value, the (weighted) surplus division solution and the prenucleolus differ completely, from an axiomatic approach the difference can be pointed out to one axiom: *consistency*, an outstanding relational property in the axiomatic method.[3]

Our starting point is the work of Hart and Mas-Colell (1989). By combining *self consistency* together with some of the following well-established properties (for two-person games): *efficiency*, *w−proportionality*, *scale invariance*, *strong aggregate monotonicity* and *equal treatment of equals*, they obtain several characterizations of both the Shapley value and the family of weighted Shapley values. Here, we are particularly interested in analysing which is the set of rules that arise from substituting *self consistency* in Hart and Mas-Colell' results by either *projected consistency* (Funaki, 1998), satisfied by ES^w, or *max consistency* (Davis and Maschler, 1965), satisfied by ν_*. Our approach follows similar lines to thoes in Chang and Hu (2007), that compare the *equal allocation of nonseparable cost value* (Moulin, 1985) with Sh in terms of consistency. Without consistency, different axiomatic comparisons of Sh and ES can be found in van den Brink (2007), Kamijo and Kongo (2012) and Casajus and Huettner (2014). To make a parallel analysis between Sh and ν_*, the papers of Orshan (1993) and Hokari (2005) are fundamental.

The paper is organized as follows. Section 2 contains some preliminaries on games. Section 3 is divided in three subsections. In Subsection 3.1, we combine *self consistency* or *projected consistency* with *(weighted) standardness*. Apart from consistency, a property that plays an important role in our work is *aggregate monotonicity* (Megiddo, 1974), that says that if the worth of the grand coalition increases, whereas the worth of all other coalitions remains the same, then everybody should be weakly better-off.[4] Because of Sh^w and ES^w distribute any variation in the worth of the grand coalition following a fix pattern reflecting some

[2]Other authors, like Kalai and Samet (1987), Monderer, Samet and Shapley (1992) or Nowak and Radzik (1995) give a more general definition of Sh^w that allows some players to have zero weight. Béal et al. (2015) consider a wide class of weighted surplus division values where any set of players is assigned to a vector of weights which may contain negative coordinates and differs for different sets.

[3]See Thomson (2011) and Thomson (2012) for essays on the consistency principle.

[4]In the literature, several notions of monotonicity have been introduced to characterize solutions on different frameworks (e.g., Kalai and Smorodinsky, 1975; Kalai, 1977; Kalai and Samet, 1985 or Thomson, 1987).

previous agreement among players, we introduce *strong regular aggregate monotonicity* which captures this idea of regularity. In Subsection 3.2, *self consistency* and *projected consistency* are considered together with *strong regular aggregate monotonicity* and *scale invariance* (for two-person games). In Subsection 3.3, we also consider the classical properties of *dummy player* and *individual rationality*. Impossibility results emerge when some of the above properties are required on the domain of all games, not only for two-person games. In Section 4, we study the compatibility of *max consistency* with the aforementioned properties. This allow us to compare the family of weighted Shapley values, the family of weighted surplus division rules and ν_*. Interestingly, new impossibility results are manifest when *max consistency* appears in scene. Finally, Appendix A contains some of the proofs of the characterization results, while Appendix B shows the independence of the properties.

2 Preliminaries

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. We assume that the set of potential players contains at least two players. A **coalition** is a non-empty finite subset of $\mathbb{N}$ and let $\mathcal{N}$ denote the set of all coalitions of $\mathbb{N}$. Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ and $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. A **transferable utility coalitional game (a game)** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a real number $v(S)$, representing what S can achieve by agreeing to cooperate, with the convention that $v(\emptyset) = 0$. For simplicity of notation, and if no confusion arises, we write $v(i), v(ij), \ldots$ instead of $v(\{i\}), v(\{i, j\}), \ldots$. By Γ we denote the class of all games.

Given $N \in \mathcal{N}$ and $\emptyset \neq S \subseteq N$, the **unanimity game** (N, u_S) associated to S is defined as $u_S(R) = 1$ if $S \subseteq R$ and $u_S(R) = 0$ otherwise. Given a game (N, v) and $\emptyset \neq N' \subset N$, the **subgame** $(N', v_{|N'})$ is defined as $v_{|N'}(S) = v(S)$ for all $S \subseteq N'$. For any two games (N, v), (N, w), and $\alpha \in \mathbb{R}$, we define the game $(N, v + w)$ as $(v + w)(S) = v(S) + w(S)$, and the game $(N, \alpha \cdot v)$ as $(\alpha \cdot v)(S) = \alpha \cdot v(S)$, for all $S \subseteq N$. The **null** game $(N, \mathbf{0})$ is defined by $\mathbf{0}(S) = 0$ for all $S \subseteq N$.

Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. Given $\emptyset \neq S \subseteq N$, $e_S \in \mathbb{R}^N$ is defined as $e_{S,i} = 1$ if $i \in S$ and $e_{S,i} = 0$ otherwise. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} = (x_i)_{i \in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$, for all $i \in N$, while $x > y$ if $x_i > y_i$, for all $i \in N$.

The set of feasible payoff vectors of (N, v) is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$, while the **preimputation set** contains the **efficient** payoff vectors, that is, $X(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$. The set of **imputations** is defined by $I(N,v) := \{x \in X(N,v) \,|\, x_i \geq v(\{i\})$ for all $i \in N\}$. The **core** of (N,v) is the set of those imputations where each coalition obtain at least its worth, that is $C(N,v) := \{x \in I(N,v) \,|\, x(S) \geq v(S)$ for all $S \subset N\}$. A game (N,v) is said to be **balanced** if $C(N,v) \neq \emptyset$.

A **solution** on a class of games $\Gamma' \subseteq \Gamma$ is a correspondence σ that associates with each game $(N,v) \in \Gamma'$ a subset $\sigma(N,v)$ of $X^*(N,v)$. A solution σ on $\Gamma' \subseteq \Gamma$ is said to be **single-valued** if $|\sigma(N,v)| = 1$ for all $(N,v) \in \Gamma'$. Note that a single-valued solution is always non-empty but not necessarily an efficient allocation. We say that a single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies **efficiency** (**E**) if all the gains from cooperation are shared among the players, that is, for all $N \in \mathcal{N}$ and all $(N,v) \in \Gamma'$, it holds $\sum_{i \in N} \sigma_i(N,v) = v(N)$. For our purposes, we introduce some well-known efficient single-valued solutions defined on Γ. Let $N \in \mathcal{N}$ and $(N,v) \in \Gamma$. The **Shapley value**, Sh, is defined by

$$Sh_i(N,v) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(|N|-|S|-1)!}{|N|!}\,(v(S \cup \{i\}) - v(S)) \text{ for all } i \in N.$$

Let $(N,v) \in \Gamma$ and let $\alpha_T = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ for all $\emptyset \neq T \subseteq N$. Then we can express the game (N,v) by a linear combination of the unanimity games as $v = \sum_{\emptyset \neq T \subseteq N} \alpha_T u_T$. The **weighted Shapley value** relative to a list of positive weights $w = (w_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$, Sh^w, is defined by

$$Sh^w(N,v) := \sum_{\emptyset \neq T \subseteq N} \alpha_T \cdot Sh^w(N,u_T),$$

where

$$Sh^w_i(N,u_T) = \begin{cases} \frac{w_i}{\sum_{j \in T} w_j} & \text{if } i \in T \\ 0 & \text{if } i \in N \setminus T \end{cases}.$$

Note that when $w_i = w_j$ for all $i, j \in \mathbb{N}$, then $Sh^w(N,v) = Sh(N,v)$.

The **equal surplus division** solution, ES, is defined by

$$ES_i(N,v) := v(i) + \frac{1}{|N|}\left(v(N) - \sum_{i \in N} v(i)\right) \text{ for all } i \in N.$$

The **weighted surplus division** solution relative to a list of positive weights $w = (w_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$, ES^w, is defined by

$$ES^w_i(N,v) := v(i) + \frac{w_i}{\sum_{j \in N} w_j}\left(v(N) - \sum_{i \in N} v(i)\right) \text{ for all } i \in N.$$

Given a list of positive weights w, ES^w can be interpreted as a two-stage rule: after assigning to every player what they can achieve for themselves alone, it distributes what is left of the gains of cooperation proportionally according to w. Note that when $w_i = w_j$ for all $i, j \in \mathbb{N}$, then $ES^w(N,v) = ES(N,v)$.

3 (Weighted) Shapley value versus (weighted) surplus division rule

The main concern of this section is to show that, from an axiomatic point of view, the consistency principle distinguishes Sh^w and ES^w. The section is organized as follows. First, we introduce the notion of consistency. In every subsection we study how consistency combines with other basic properties: Subsection 3.1 considers *weighted standardness*. Subsection 3.2 focus on *strong regular aggregate monotonicity* together with *scale invariance*. Finally, Subsection 3.3 is devoted to *strong regular aggregate monotonicity*, the *dummy player* property and *individual rationality*. For a clearer presentation, the proofs of the characterization results that make use of technical arguments are consigned in Appendix A. Appendix B contains the independence of the properties of the characterizations.

Consistency is a sort of internal stability requirement that relates the solution of a game to the solution of a reduced game that results when some agents leave. The different ways in which the agents that remain evaluate the possible coalitions give rise to different notions of reduced game. In this section, we deal with two ways of reducing a game: the **self reduced game** and the **projected reduced game**. The terminology is taken from Thomson (2003).

Definition 1. *Let σ be a single-valued solution, $N \in \mathcal{N}$, $(N,v) \in \Gamma$, and $\emptyset \neq N' \subset N$. The* **self reduced game** *relative to N' at σ is the game $\left(N', r^{N'}_{S,\sigma}(v)\right)$ defined by*

$$r^{N'}_{S,\sigma}(v)(R) := \begin{cases} 0 & \text{if } R = \emptyset, \\ v(R \cup N \setminus N') - \sum_{i \in N \setminus N'} \sigma_i(R \cup N \setminus N', v_{|R \cup N \setminus N'}) & \text{if } \emptyset \neq R \subseteq N'. \end{cases}$$

Definition 2. *Let $N \in \mathcal{N}$, $(N,v) \in \Gamma$, $x \in \mathbb{R}^N$ and $\emptyset \neq N' \subset N$. The* ***projected reduced game*** *relative to N' at x is the game $\left(N', r^{N'}_{P,x}(v)\right)$ defined by*

$$r^{N'}_{P,x}(v)(R) := \begin{cases} v(R) & \text{if } R \subset N', \\ v(N) - x(N \setminus N') & \text{if } R = N'. \end{cases}$$

In the self reduced game (relative to N' at σ), the worth of a coalition $R \subseteq N'$ is determined under the assumption that R joins all members of $N \setminus N'$, provided they are paid according to σ in the subgame associated to $R \cup (N \setminus N')$. In the projected reduced game (relative to N' at x), when players in $N \setminus N'$ leave the game, for a proper subcoalition $R \subset N'$ cooperation is no longer possible with them.

The following notions of consistency rely on the above definitions of reduced game. A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Self consistency (SC)**: if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$ and all $\emptyset \neq N' \subset N$, then $\left(N', r^{N'}_{S,\sigma}(v)\right) \in \Gamma'$ and $\sigma(N,v)_{|N'} = \sigma\left(N', r^{N'}_{S,\sigma}(v)\right)$.

- **Projected consistency** (**PC**): if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$, all $\emptyset \neq N' \subset N$ and $x = \sigma(N,v)$, then $\left(N', r^{N'}_{P,x}(v)\right) \in \Gamma'$ and $x_{|N'} = \sigma\left(N', r^{N'}_{P,x}(v)\right)$.

Self consistency and *projected consistency* state that in the self reduced game and in the projected reduced game, respectively, the original agreement should be confirmed.

3.1 Consistency and weighted standardness

For the two-agent case, many solutions in the literature coincide with the standard solution or its weighted generalizations.

The **standard solution**, ST, is defined as follows: for all $N = \{i,j\} \in \mathcal{N}$ and all $(N,v) \in \Gamma$,

$$ST_i(N,v) := v(i) + \frac{1}{2}\left(v(N) - v(i) - v(j)\right),$$

$$ST_j(N,v) := v(j) + \frac{1}{2}\left(v(N) - v(i) - v(j)\right).$$

The **weighted standard solution** relative to a list of positive weights $w = (w_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$, ST^w, is defined as follows: for all $N = \{i,j\} \in \mathcal{N}$ and all $(N,v) \in \Gamma$,

$$ST^w_i(N,v) := v(i) + \frac{w_i}{w_i + w_j}\left(v(N) - v(i) - v(j)\right),$$

$$ST^w_j(N,v) := v(j) + \frac{w_j}{w_i + w_j}\left(v(N) - v(i) - v(j)\right).$$

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Standardness** (**ST**): if for all $N = \{i,j\} \in \mathcal{N}$ and all $(N,v) \in \Gamma'$, it holds $\sigma(N,v) = ST(N,v)$.

- w**-proportionality** (w−**P**) (w.r.t. a list of positive weights $w \in \mathbb{R}^{\mathbb{N}}_{++}$): if for all $N = \{i,j\} \in \mathcal{N}$ and all $(N,v) \in \Gamma'$, it holds $\sigma(N,v) = ST^w(N,v)$.

A solution satisfies *Standardness* if for any two player game (N,v) assigns to each player what they can achieve for themselves and equally divides the surplus $v(N) - v(i) - v(j)$, while it satisfies *w-proportionality* if shares this surplus proportionally according to a given list of positive weights w. Note that *standardness* is a particular case of *w-proportionality* where all the players have the same weight. Sh and ES satisfy *standarness* and, for a given $w \in \mathbb{R}^{\mathbb{N}}_{++}$, Sh^w and ES^w satisfy *w-proportionality*.

On the domain of all games, and for a given $w \in \mathbb{R}^{\mathbb{N}}_{++}$, Sh^w is the unique single-valued solution that satisfies *self consistency* and *w-proportionality* (Theorem 5.7 in Hart and Mas-Colell, 1989). In addition, Theorem B in the same paper states that Sh is the unique single-valued solution satisfying *self consistency* and *standardness*.

Interestingly, ES^w and ES can be characterized by replacing *self consistency* with *projected consistency.*[5]

Theorem 1. *Let $w \in \mathbb{R}^{\mathbb{N}}_{++}$ be a list of positive weights. The weighted surplus division value, ES^w, is the unique single-valued solution on Γ satisfying projected consistency and $w-$proportionality (w.r.t. $w \in \mathbb{R}^{\mathbb{N}}_{++}$).*

By using *standardness* instead of *w-proportionality* it follows the next corollary, already stated by van den Brink et al. (2016) and implicitly suggested by Driessen and Funaki (1997).

Corollary 1. *The equal surplus division value, ES, is the unique single-valued solution on Γ satisfying projected consistency and standardness.*

The above characterizations are summarized in the following scheme:

$$ES^w \equiv \mathbf{PC} + w{-}\mathbf{P} \text{ (Th. 1).}$$

$$Sh^w \equiv \mathbf{SC} + w{-}\mathbf{P} \text{ (Th. 5.7 in Hart and Mas-Colell, 1989).}$$

$$ES \equiv \mathbf{PC} + \mathbf{ST} \text{ (Cor. 1).}$$

$$Sh \equiv \mathbf{SC} + \mathbf{ST} \text{ (Th. B in Hart and Mas-Colell, 1989).}$$

3.2 Consistency, monotonicity and scale invariance

In this subsection, our goal is to replace the prescriptive property of *w-proportionality* by *scale invariance* and a strong version of *aggregate monotonicity.*

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies:

- **scale invariance (SI)**: if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $\alpha > 0$ and all $d \in \mathbb{R}^N$, if (N, w) is such that $w(S) = \alpha \cdot v(S) + d(S)$ for all $S \subseteq N$, then $\sigma(N, w) = \alpha \cdot \sigma(N, v) + d$.

- **Strong aggregate monotonicity (SAM)**: if for all $N \in \mathcal{N}$ and all (N, v), $(N, v') \in \Gamma'$ with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$, it holds $\sigma(N, v) < \sigma(N, v')$.

- **Equal aggregate monotonicity (EAM)**: if for all $N \in \mathcal{N}$ and all (N, v), $(N, v') \in \Gamma'$ with $v(S) = v'(S)$ for all $S \subset N$, it holds

$$\sigma(N, v') - \sigma(N, v) = \left(\frac{v'(N) - v(N)}{|N|}, \ldots, \frac{v'(N) - v(N)}{|N|} \right).$$

[5]Here, it is worth to remark that the class of weighted surplus division solutions introduced by Béal et al. (2016) is not projected consistent.

Scale invariance is a classical invariant requirement w.r.t. changes in scale that are comparable with positive affine transformations. *Strong aggregate monotonicity* states that everybody is strictly better off whenever the worth of the grand coalition increases and the worth of every other coalition remains unchanged. Remarkably, both Sh and ES satisfy *equal aggregate monotonicity*, which is a much stronger property imposing that agents share equally the raise of the worth of the grand coalition.[6]

According to *equal aggregate monotonicity*, any set of players N agree on distribute equally any amount $t \in \mathbb{R}$, representing the difference of the worth of the grand coalition between two games. However, there are many other monotonic ways of distributing t. A monotone path is just a complete list of these monotonic agreements.

Definition 3. *A (strict) monotone path is a function $f : \mathcal{N} \times \mathbb{R} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ satisfying the following conditions: for all $N \in \mathcal{N}$ and all $t \in \mathbb{R}$,*

1. $f(N, 0) = (0, \ldots, 0) \in \mathbb{R}^N$,
2. $f(N, t) \in \mathbb{R}^N$ *and* $\sum_{i \in N} f_i(N, t) = t$,
3. *if* $t' \in \mathbb{R}$ *is such that* $t' > t$, *then* $f_i(N, t')\ (>)\ \geq f_i(N, t)$ *for all* $i \in N$.

Note that a monotone path assigns non-negative (non-positive) vectors to positive (negative) real numbers.

Let $\mathcal{F}_{mon}$ denote the class of monotone paths. Examples of functions in $\mathcal{F}_{mon}$ that will be used along the paper are:

1. For all $N \in \mathcal{N}$, all $t \in \mathbb{R}$ and all $i \in N$, define $\bar{f}_i(N, t) = \frac{t}{|N|}$.

   $\bar{f}$ distributes t equally among players in N.

2. Let $w \in \mathbb{R}^{\mathbb{N}}_{++}$ be a list of positive weights. For all $N \in \mathcal{N}$, all $t \in \mathbb{R}$ and all $i \in N$, define $f^w_i(N, t) = \frac{w_i}{\sum\limits_{j \in N} w_j} \cdot t$.

   f^w distributes t among players in N proportionally according to their weights w.

3. Let π be a permutation on $\mathbb{N}$. For all $N \in \mathcal{N}$ and all $t \in \mathbb{R}$, define $f^\pi(N, t) = t \cdot e_{\{j\}}$, being $j \in N$ such that $\pi(j) \geq \pi(i)$ for all $i \in N$.

   f^π assigns all the amount t to the last player in N according to π.

Instances of strict monotone path are $\bar{f}$ and f^w, while f^π is not.

By using the notion of a strict monotone path, we introduce *strong regular aggregate monotonicity*.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

[6] A property related with *equal aggregate monotonicity* is *weak fairness* (van den Brink et al., 2016). This property together with *efficiency* imply *equal aggregate monotonicity*.

- **Strong regular aggregate monotonicity** (**SRAM**): if there exists a strict monotone path $f \in \mathcal{F}_{mon}$ such that, for all $N \in \mathcal{N}$ and all $(N, v), (N, v') \in \Gamma'$ with $v(S) = v'(S)$ for all $S \subset N$, it holds $\sigma(N, v') - \sigma(N, v) = f(N, v'(N) - v(N))$.

Strong regular aggregate monotonicity requires that any set of players $N \in \mathcal{N}$ reaches an agreement (which can be different for different sets) on how to distribute monotonically any change in the worth of the grand coalition. Clearly, *strong regular aggregate monotonicity* implies *strong aggregate monotonicity* and it is implied by *equal aggregate monotonicity.*

Theorem C in Hart and Mas-Colell (1989) states that Sh can be characterized by means of *self consistency* and, for two-person games, *efficiency*, *scale invariance* and *strong aggregate monotonicity.*[7] Unfortunaty, it is still an open question if replacing *self consistency* by *projected consistency* we reach a characterization of the family of weighted surplus division solutions. However, it turns out that this class of single-valued solutions can be characterized by means of *projected consistency* and, for two-person games, *scale invariance* and *strong regular aggregate monotonicity.* To prove this result (Theorem 2 (*i*)) we make use of three lemmas (Lemma 2, Lemma 3 and Lemma 4 stated in Appendix A) that show that if a single-valued solution σ satisfies these three properties, then there exists a list of positive weights $w \in \mathbb{R}^{\mathbb{N}}_{++}$ such that σ also satisfies *w-proportionality* (w.r.t. w). It is straightforward to check that these lemmas also hold by replacing *projected consistency* with *self consistency.* This fact, together with Theorem 5.7 in Hart and Mas-Colell (1989), leads to a parallel characterization for the family of weighted Shapley values ((Theorem 2 (*ii*)).

Theorem 2. *Let σ be a single valued solution on Γ that satisfies scale invariance and strong regular aggregate monotonicity for two-person games. Then,*

(i) σ satisfies projected consistency if and only if there exists a list of positive weights $w \in \mathbb{R}^{\mathbb{N}}_{++}$ such that $\sigma = ES^w$.

(ii) σ satisfies self consistency if and only if there exists a list of positive weights $w \in \mathbb{R}^{\mathbb{N}}_{++}$ such that $\sigma = Sh^w$.

A direct consequence of Theorem 2 is a characterization of ES and Sh by replacing *strong regular aggregate monotonicity* with *equal aggregate monotonicity.*

Corollary 2. *Let σ be a single valued solution on Γ that satisfies scale invariance and equal aggregate monotonicity for two-person games. Then,*

(i) σ satisfies projected consistency if and only if $\sigma = ES$.

[7] In Hart and Mas-Colell (1989), *scale invariance* and *strong aggregate monotonicity* are named *TU-invariance* and *monotonicity*, respectively.

(ii) σ *satisfies self consistency if and only if* $\sigma = Sh$.

Remark 1. *Let us point out two facts:*

1. *All the characterization results and the non-redundancy of the properties stated in this section remains valid even if scale invariance, strong regular aggregate monotonicity and equal aggregate monotonicity are formulated on the domain of all games, not only for two-person games.*

2. *In view of the fact that Lemma 2 (see Appendix A), that connects scale invariance for two person games and projected consistency with efficiency, also holds by imposing either self consistency or max consistency instead of projected consistency, if the universe of potential players contains, at least, two players, efficiency can be dropped in Theorem C in Hart and Mas-Colell (1989). When scale invariance is required for all games, not only for two-person games, Peleg and Sudhölter (2007) (Lemma 6.2.11 and Lemma 8.3.8) already note that either scale invariance together with self consistency and scale invariance together with max consistency imply efficiency.*

Next, we outline the above characterizations:

$$\{ES^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\} \equiv \textbf{PC} + \{\textbf{SI} + \textbf{SRAM}\} \text{ (2-person games) (Th. 2 } (i)),$$

$$\{Sh^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\} \equiv \textbf{SC} + \{\textbf{SI} + \textbf{SRAM}\} \text{ (2-person games) (Th. 2 } (ii)).$$

$$ES \equiv \textbf{PC} + \{\textbf{SI} + \textbf{EAM}\} \text{ (2-person games) (Cor. 2 } (i)),$$

$$Sh \equiv \textbf{SC} + \{\textbf{SI} + \textbf{EAM}\} \text{ (2-person games) (Cor. 2 } (ii)).$$

3.3 Consistency, monotonicity and dummy player property or individual rationality

Here, we compare axiomatically Sh with ES by imposing, for two-person games, either *equal aggregate monotonicity* and *dummy player property* or *equal aggregate monotonicity* and *individual rationality* instead of *strong regular aggregate monotonicity* and *scale invariance.* The *dummy player property* says that players that do not contribute anything (except their individual worth) should receive their individual worth. *Individual rationality* means that no single player can improve the payoff proposed by the solution without cooperation.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Dummy player property** (**DP**) if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and $i \in N$, if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus \{i\}$, then $\sigma_i(N, v) = v(i)$.

- **Individual rationality** (**IR**) if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$ with $I(N, v) \neq \emptyset$, it holds $\sigma_i(N, v) \geq v(i)$ for all $i \in N$.

Taking into account that *standardness* is equivalent to *dummy player property* and *equal aggregate monotonicity* or *individual rationality* and *equal aggregate monotonicity* (for two-player games) (see Lemma 6 in Appendix A), by Theorem B in Hart and Mas-Colell (1989) and Corollary 1 we obtain, respectively, the following characterizations of Sh and ES.

Theorem 3. *Let σ be a single-valued solution on Γ that satisfies equal aggregate monotonicity and dummy player property for two-person games. Then,*

(i) σ satisfies self consistency if and only if $\sigma = Sh$.

(ii) σ satisfies projected consistency if and only if $\sigma = ES$.

Theorem 4. *Let σ be a single-valued solution on Γ that satisfies equal aggregate monotonicity and individual rationality for two-person games. Then,*

(i) σ satisfies self consistency if and only if $\sigma = Sh$.

(ii) σ satisfies projected consistency if and only if $\sigma = ES$.

Remark 2. *Two observations on the compatibility of the above properties on the universal domain of games:*

1. *Since Sh satisfies equal aggregate monotonicity and the dummy player property, Theorem 3 (i) also holds on the universal domain of games. On the contrary, Theorem 3 (ii) does not. Indeed, suppose there exists a single-valued solution σ on Γ satisfying projected consistency, equal aggregate monotonicity and dummy player property on the domain of all games. Then, it must coincides with ES. But ES fails to satisfy dummy player property for games with an arbitrary number of players, which leads to a contradiction.*

2. *In a similar way, Theorem 4 (i) is not true on the domain of all games, since equal aggregate monotonicity and individual rationality together characterize ES (see Theorem 1 in Calleja and Llerena, 2016), which fails to satisfies self consistency. On the contrary, Theorem 4 (ii) holds on the domain of all games since ES satisfies equal aggregate monotonicity and individual rationality, but the properties are redundant.*

Theorem 5. *There is no single-valued solution on Γ that satisfies*

(i) Projected consistency, equal aggregate monotonicity and dummy player property.

(ii) Self consistency, equal aggregate monotonicity and individual rationality.

We end summarizing the above results:

$$Sh \equiv \mathbf{SC} + \{\mathbf{DP} + \mathbf{EAM}\} \text{ (2-person games) (Th. 3 }(i)).$$

$$ES \equiv \mathbf{PC} + \{\mathbf{DP} + \mathbf{EAM}\} \text{ (2-person games) (Th. 3 }(ii)),$$

$$Sh \equiv \mathbf{SC} + \{\mathbf{IR} + \mathbf{EAM}\} \text{ (2-person games) (Th. 4 }(i)),$$

$$ES \equiv \mathbf{PC} + \{\mathbf{IR} + \mathbf{EAM}\} \text{ (2-person games) (Th. 4 }(ii)),$$

PC + **DP** + **EAM** are **incompatible** (Th. 5 (i)),

SC + **IR** + **EAM** are **incompatible** (Th. 5 (ii)).

4 Remarks on max consistency

The aim of this section is to study how well *max consistency* combines with some of the properties we have used in the previous section.

Definition 4. *Let $N \in \mathcal{N}$, $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$ and $\emptyset \neq N' \subset N$. The* **max reduced game** *relative to N' at x is the game $\left(N', r^{N'}_{M,x}(v)\right)$ defined by*

$$r^{N'}_{M,x}(v)(R) := \begin{cases} 0 & \text{if } R = \emptyset, \\ \max\limits_{Q \subseteq N \setminus N'} \{v(R \cup Q) - x(Q)\} & \text{if } \emptyset \neq R \subset N', \\ v(N) - x(N \setminus N') & \text{if } R = N'. \end{cases}$$

In the max reduced game (relative to N' at x), the worth of a coalition $R \subset N'$ is determined under the assumption that R can choose the best partners in $N \setminus N'$, provided that it pays them according to x.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Max consistency** (**MC**): if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $\emptyset \neq N' \subset N$, and $x = \sigma(N, v)$, then $\left(N', r^{N'}_{M,x}(v)\right) \in \Gamma'$ and $x_{|N'} = \sigma\left(N', r^{N'}_{M,x}(v)\right)$.

A solution satisfies *max consistency* if it assigns the same payoff to players in both the original game and the max reduced game.

A well-known single-valued solution satisfying *max consistency* is the prenucleolus. Let $N \in \mathcal{N}$ and $(N, v) \in \Gamma$. With any preimputation $x \in X(N, v)$ we associate the vector of all excesses $e(S, x) = v(S) - x(S)$, $\varnothing \neq S \subset N$, the components of which are non-increasingly ordered. The **prenucleolus**, ν_*, is the preimputation that minimizes with respect to the lexicographic order[8] the vector of excesses over the set of preimputations.

The next two properties will be a key tool in order to compare axiomatically Sh^w, ES^w and ν_*.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

[8]Given two vector $x, y \in \mathbb{R}^N$, we say that $x \leq_{lex} y$ if either $x = y$, or $x_1 < y_1$ or there exists $k \in \{2, \ldots, |N|\}$ such that $x_i = y_i$ for all $1 \leq i \leq k-1$ and $x_k < y_k$.

- **Weighted proportionality** (**WP**) if there exists a list of positive weights $w \in \mathbb{R}^{\mathbb{N}}_{++}$ such that σ satisfies *w-proportionality* (w.r.t. w).

The property of 2− *Weighted standardness* (Hokari, 2005) is closely related with *weighted proportionality.*

Given a pair of agents $N' = \{i, j\} \in \mathcal{N}$, let $\Delta^{N'} := \{(a, b) \in \mathbb{R}^{N'}_{+} \mid a + b = 1\}$. For each $\alpha \in \Delta^{N'}$, consider the following single-valued solution on the domain of all games with set of players N'. The **weighted standard solution relative to** α, ST^{α}, is defined as follows: for all $(N', v) \in \Gamma$,

$$ST^{\alpha}_i(N', v) := v(i) + \alpha_i(v(N') - v(i) - v(j)),$$

$$ST^{\alpha}_j(N', v) := v(j) + \alpha_j(v(N') - v(i) - v(j)).$$

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- 2−**Weighted standardness** (2−**WS**) (Hokari, 2005) if

(a) for each pair of agents $N' = \{i, j\} \in \mathcal{N}$, there exists $\alpha \in \Delta^{N'}$ such that, for all $(N', v) \in \Gamma$, $\sigma(N', v) = ST^{\alpha}(N', v)$; and

(b) for all distinct three players $\{i, j, k\} \in \mathcal{N}$, if $\alpha = (\alpha_i, \alpha_j)$, $\beta = (\beta_i, \beta_k)$ and $\gamma = (\gamma_j, \gamma_k)$ are such that $\sigma(\{i, j\}, v) = ST^{\alpha}(\{i, j\}, v)$ for all $(\{i, j\}, v) \in \Gamma$, $\sigma(\{i, k\}, v) = ST^{\beta}(\{i, k\}, v)$ for all $(\{i, k\}, v) \in \Gamma$, and $\gamma(\{j, k\}, v) = ST^{\gamma}(\{j, k\}, v)$ for all $(\{j, k\}, v) \in \Gamma$, then $\alpha_i \leq \alpha_j$ and $\gamma_j \leq \gamma_k$ imply $\beta_i \leq \beta_k$.

Note that *weighted proportionality* implies 2-*weighted standardness*, but the reverse implication does not hold.

Theorem 1 in Hokari (2005) states that, on the domain of all games, the prenucleolus can be characterized by means of *efficiency*, *zero-independence*,[9] 2−*weighted standardness* and *max consistency*. Since *standardness* implies 2−*weighted standardness*, *scale invariance* implies *zero-independence* and *scale invariance* with *max consistency* together imply *efficiency* (see Remark 1 (ii)), Theorem 1 in Hokari (2005) can be rewritten in terms of *max consistency*, *scale invariance* and *standardness*. Interestingly, this reformulation of Hokari's result together with the characterizations of Sh and ES provided, respectively, by Theorem B in Hart and Mas-Colell (1989) and Corollary 1, lead to the following comparison:

$$\nu_* \equiv \mathbf{MC} + \mathbf{ST} + \mathbf{SI},$$

$$Sh \equiv \mathbf{SC} + \mathbf{ST} \text{ (Th. B in Hart and Mas-Colell, 1989)},$$

[9]A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies **zero-independence** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $d \in \mathbb{R}^N$, if (N, w) is such that $w = v + d$, then $\sigma(N, w) = \sigma(N, v) + d$ where $w(S) = v(S) + d(S)$ for all $S \subseteq N$. Note that *zero-independence* is a particular case of *scale invariance* for $\alpha = 1$.

$$ES \equiv \textbf{PC} + \textbf{ST} \text{ (Cor. 1).}$$

By replacing *standardness* with *weighted proportionality* we obtain an additional comparative analysis. Proposition 1 in Hokari (2005) states that, on the domain of all games, *efficiency*, $2-$*weighted standardness* and *max consistency* jointly imply *standardness*. Since *weighted proportionallity* implies $2-$*weighted standardness*, ν_* can be characterized in terms of *weighted proportionality*, *scale invariance* and *max consistency*. On the other hand, *max consistency* together with *scale invariance* imply *efficiency* (see Remark 1 (ii)). This, together with Theorem 1 and Theorem 5.7 in Hart and Mas-Colell (1989) lead to the following comparison:

$$\nu_* \equiv \textbf{MC} + \textbf{WP} + \textbf{SI},$$

$$\{Sh^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\} \equiv \textbf{SC} + \textbf{WP},$$

$$\{ES^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\} \equiv \textbf{PC} + \textbf{WP}.$$

According to this axiomatic analysis, consistency distinguishes these three solution concepts since all of them satisfy *scale invariance*.

Theorem 2 compares axiomatically the family of weighted surplus division rules with the family of weighted Shapley values by means of *scale invariance* and *strong regular aggregate monotonicity* (for two-person games) together with either *projected consistency* or *self consistency*, respectively. These two results also hold imposing *scale invariance* and *strong regular aggregate monotonicity* on the domain of all games (see Remark 1 (i)). Unfortunately, we find an impossibility if we consider *max consistency*.

Theorem 6. *There is no single valued solution on* Γ *that satisfies max consistency, scale invariance and strong regular aggregate monotonicity.*

In summary,

$$\textbf{MC} + \textbf{SI} + \textbf{SRAM} \text{ are } \textbf{incompatible} \text{ (Th. 6)},$$

$$\{ES^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\} \equiv \textbf{PC} + \textbf{SI} + \textbf{SRAM} \text{ (Th. 2 (i))},$$

$$\{Sh^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\} \equiv \textbf{SC} + \textbf{SI} + \textbf{SRAM} \text{ (Th. 2 (ii))}.$$

As commented in Subsection 3.3, *equal aggregate monotonicity*, the *dummy player* property and *self consistency* characterize Sh (see Remark 2 (i)), but compatibility is lost if we deal with *projected consistency* (Theorem 5 (i)). This impossibility result also holds by considering *max consistency*. In addition, *max consistency*, *individual rationality* and *equal aggregate monotonicity* also are incompatible since *individual rationality* and *equal aggregate monotonicity* together characterize ES (see Theorem 1 in Calleja and Llerena, 2016), which fails to satisfies *max consistency*.

Theorem 7. *There is no single-valued solution on* Γ *that satisfies*

(i) Max consistency, equal aggregate monotonicity and dummy player property.

(ii) Max consistency, equal aggregate monotonicity and individual rationality.

In summary,

$$Sh \equiv \mathbf{SC} + \mathbf{DP} + \mathbf{EAM}.$$

PC + **DP** + **EAM** are **incompatible** (Th. 5 (i)),

MC + **DP** + **EAM** are **incompatible** (Th. 7 (i)),

SC + **IR** + **EAM** are **incompatible** (Th. 5 (ii)),

MC + **IR** + **EAM** are **incompatible** (Th. 7 (ii)).

Finally, we introduce *Equal treatment of equals*, which states that if two players contribute equal amounts to all coalitions, their payoff should be equal.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Equal treatment of equals** (**ETE**): if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and $i, j \in N$, if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, then $\sigma_i(N, v) = \sigma_j(N, v)$.

It is well-known that *efficiency*, *scale invariance* and *equal treatment of equals* for two-person games are equivalent to *standardness*. Moreover, Sh and ES satisfies these properties for any game. Since, *scale invariance* together with either *self consistency* or *projected consistency* imply *efficiency* (see Remark 1 (i)), Theorem B' in Hart and Mas-Colell (1989) and Corollary 4.4. in Driessen and Funaki (1997) can be reformulated in terms of *scale invariance* and *equal treatment of equals* together with either *self consistency* or *projected consistency*, respectively.[10] Interestingly, Orshan (1993) characterizes the prenucleolus making use of *scale invariance*, *equal treatment of equals* and *max consistency*. This lead to the following comparison:

$$\nu_* \equiv \mathbf{MC} + \mathbf{ETE} + \mathbf{SI} \text{ (Orshan, 1993)},$$

$$Sh \equiv \mathbf{SC} + \mathbf{ETE} + \mathbf{SI},$$

$$ES \equiv \mathbf{PC} + \mathbf{ETE} + \mathbf{SI} \text{ (Driessen and Funaki, 1997)}.$$

[10] In Driessen and Funaki (1997), *efficiency* is included in their definition of single-valued solution.

Appendix A

Before proving the characterization results, we need some previous results.

The next lemma will be of help to prove Theorem 1.

Lemma 1. *Let σ be a single-valued solution on Γ satisfying projected consistency and $w-$proportionality (w.r.t. $w \in \mathbb{R}^{\mathbb{N}}_{++}$). Then, σ satisfies efficiency.*

Proof Let σ be a single-valued solution on Γ satisfying **PC** and w-**P** w.r.t. $w \in \mathbb{R}^{\mathbb{N}}_{++}$. Let $(\{i\}, v)$ be a one-person game and, for some $j \in \mathbb{N}\backslash\{i\}$, consider the game $(\{i,j\}, v')$ defined by $v'(i) = v'(ij) = v(i)$ and $v'(j) = 0$. By w-**P**, $\sigma_i(\{i,j\}, v') = v(i)$ and $\sigma_j(\{i,j\}, v') = 0$. It is easy to check that $(\{i\}, v) = \left(\{i\}, r^{\{i\}}_{P,x}(v')\right)$ being $x = \sigma(\{i,j\}, v')$. By **PC**, $\sigma(\{i\}, v) = v(i)$ which means that σ satisfies efficiency for one-person games. Let $N \in \mathcal{N}$ with $|N| \geq 2$, $(N, v) \in \Gamma$ and $i \in N$. Then, efficiency for one-person game implies $\sigma_i\left(\{i\}, r^{\{i\}}_{F,x}(v)\right) = r^{\{i\}}_{F,x}(v)(i) = v(N) - \sum_{j \in N\backslash\{i\}} \sigma_j(N, v)$, where $x = \sigma(N, v)$. By **PC**, $\sigma_i(N, v) = \sigma_i\left(\{i\}, r^{\{i\}}_{F,x}(v)\right)$ and thus $\sigma_i(N, v) = v(N) - \sum_{j \in N\backslash\{i\}} \sigma_j(N, v)$, which proves **E**. □

To prove Theorem 2, the following lemmas will be of help.

Lemma 2. *Let σ be a single-valued solution on Γ satisfying scale invariance for two-person games and projected consistency. Then, σ satisfies efficiency.*

Proof Let σ be a single-valued solution on Γ that satisfies **SI** for two-person games and **PC**. Let $(\{i,j\}, \mathbf{0})$ be the null game. Then, by **SI** (for two-person games) we have $\sigma(\{i,j\}, \mathbf{0}) = \sigma(\{i,j\}, 2 \cdot \mathbf{0}) = 2 \cdot \sigma(\{i,j\}, \mathbf{0})$ and, consequently, $\sigma(\{i,j\}, \mathbf{0}) = (0, 0)$.

Let $(\{i,j\}, v)$ be a game such that $v(ij) = v(i) + v(j)$. Then, by **SI** (for two-person games) we have

$$\sigma(\{i,j\}, v) = \sigma(\{i,j\}, 1 \cdot \mathbf{0} + (v(i), v(j)) = 1 \cdot \sigma(\{i,j\}, \mathbf{0}) + (v(i), v(j)) = (v(i), v(j)). \tag{1}$$

Now, let $(\{i\}, v)$ be a one person game and, for some $j \in \mathbb{N}\backslash\{i\}$, consider the game $(\{i,j\}, v')$ defined by $v'(i) = v'(ij) = v(i)$ and $v'(j) = 0$. Since $v'(ij) = v'(i) + v'(j)$, from (1) it comes that $\sigma_i(\{i,j\}, v') = v(i)$ and $\sigma_j(\{i,j\}, v') = 0$. It is easy to check that $(\{i\}, v) = \left(\{i\}, r^{\{i\}}_{P,x}(v')\right)$ being $x = \sigma(\{i,j\}, v')$. By **PC**, $\sigma(\{i\}, v) = v(i)$ which implies efficiency for one-person games and, consequently, **E** (following the same argument as in the last part of the proof of Lemma 1). □

Remark 3. *Lemma 1 and Lemma 2 hold for any consistency property in which the worth of the grand coalition in the reduced game is the difference between the worth of the grand coalition in the original game and the payoff received, according to the solution, by players leaving the game. In particular, they hold for both self consistency and max consistency.*

Lemma 3. *Let σ be a single-valued solution on Γ satisfying scale invariance and strong regular aggregate monotonicity for two-person games. Then, for all $\{i,j\} \in \mathcal{N}$ it holds*

1. $\sigma(\{i,j\}, u_{\{i,j\}}) > (0,0)$
2. $\sigma(\{i,j\}, u_{\{i,j\}}) = -\sigma(\{i,j\}, -u_{\{i,j\}})$

Proof Let σ be a single-valued solution on Γ that satisfies **SI** and **SRAM** for two-person games. Let $N = \{i,j\}$ and consider the associated unanimity game (N, u_N). By **SRAM** (for two-person games), there exists a strict monotone path $f \in \mathcal{F}_{mon}$ such that $\sigma(N, u_N) = \sigma(N, \mathbf{0}) + f(N, 1)$. Similarly, $\sigma(N, -u_N) = \sigma(N, \mathbf{0}) + f(N, -1)$. By **SI** (for two-person games), $\sigma(N, \mathbf{0}) = (0,0)$ and thus $\sigma(N, u_N) = f(N, 1)$ and $\sigma(N, -u_N) = f(N, -1)$. Hence, $\sigma(N, u_N) > (0,0)$ which proves statement 1. Moreover,

$$\begin{aligned}
\sigma(N, u_N) + \sigma(N, -u_N) &= f(N,1) + f(N,-1) \\
&= f(N, (u_N - \mathbf{0})(N)) + f(N, (\mathbf{0} - u_N)(N)) \\
&= \sigma(N, u_N) - \sigma(N, \mathbf{0}) + \sigma(N, \mathbf{0}) - \sigma(N, u_N) \\
&= (0,0),
\end{aligned}$$

where the last but one inequality follows from **SRAM** (for two-person games). This proves statement 2. □

Lemma 4. *Let σ be a single-valued solution on Γ satisfying projected consistency and, for two-person games, scale invariance and strong regular aggregate monotonicity. Let $N \in \mathcal{N}$ with $|N| = 3$. Then, for all $k, s \in N$ it holds*

$$\frac{\sigma_k(N, u_N)}{\sigma_s(N, u_N)} = \frac{\sigma_k\left(\{k,s\}, u_{\{k,s\}}\right)}{\sigma_s\left(\{k,s\}, u_{\{k,s\}}\right)}. \tag{2}$$

Proof Let σ be a single-valued solution on Γ that satisfies **PC** and, for two-person games, **SI** and **SRAM**. Let $N \in \mathcal{N}$ with $|N| = 3$ and $k, s \in N$. Let us denote $x = \sigma(N, u_N)$. By the definition of projected reduced game, **PC**, **SI** (for two-person games) and Lemma 3 (2) we obtain, for all pair of agents $k, s \in N$,

$$\begin{aligned}
\sigma_{|\{k,s\}}(N, u_N) &= \sigma\left(\{k,s\}, r_{P,x}^{\{k,s\}}(u_N)\right) \\
&= \sigma\left(\{k,s\}, \left(r_{P,x}^{\{k,s\}}(u_N)(ks)\right) \cdot u_{\{k,s\}}\right) \\
&= \left(r_{P,x}^{\{k,s\}}(u_N)(ks)\right) \cdot \sigma\left(\{k,s\}, u_{\{k,s\}}\right).
\end{aligned} \tag{3}$$

By Lemma 3 (1), $\sigma\left(\{k,s\}, u_{\{k,s\}}\right) > (0,0)$, which implies that $\sigma_k(N, u_N)$ and $\sigma_s(N, u_N)$ have the same sign. By **E**, $\sum_{i \in N} \sigma_i(N, u_N) = 1$ and thus $\sigma_i(N, u_N) > 0$ for all $i \in N$. Finally, from (3) if follows (2). □

Remark 4. *Lemma 4 holds for both self consistency and max consistency. For self consistency the proof follows the same lines as for projected consistency.*

The next result extends Lemma 4 when we replace *projected consistency* by *max consistency*, and it will be of help to prove Theorem 6.

Lemma 5. *Let σ be a single-valued solution on Γ satisfying max consistency and, for two-person games, scale invariance and strong regular aggregate monotonicity. Let $N \in \mathcal{N}$ with $|N| = 3$. Then, for all $k, s \in N$ it holds*

$$\frac{\sigma_k(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(k)}{\sigma_s(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(s)} = \frac{\sigma_k\left(\{k,s\},u_{\{k,s\}}\right)}{\sigma_s\left(\{k,s\},u_{\{k,s\}}\right)}, \tag{4}$$

where $x = \sigma(N, u_N)$.

Proof Let σ be a single-valued solution on Γ that satisfies **SI** and **SRAM** for two-person games, and **MC**. Let $N \in \mathcal{N}$ with $|N| = 3$. Let us denote $x = \sigma(N, u_N)$. By **MC**, **SI** (for two-person games) and Lemma 3 (2) we obtain, for all pair of agents $k, s \in N$,

$$\begin{aligned}
\sigma_{|\{k,s\}}(N,u_N) &= \sigma_k\left(\{k,s\}, r^{\{k,s\}}_{M,x}(u_N)\right) \\
&= \left(r^{\{k,s\}}_{M,x}(u_N)(ks) - r^{\{k,s\}}_{M,x}(u_N)(k) - r^{\{k,s\}}_{M,x}(u_N)(s)\right)\cdot \sigma\left(\{k,s\},u_{\{k,s\}}\right) \\
&+ \left(r^{\{k,s\}}_{M,x}(u_N)(k), r^{\{k,s\}}_{M,x}(u_N)(s)\right).
\end{aligned} \tag{5}$$

By Lemma 3 (1) $\sigma\left(\{k,s\},u_{\{k,s\}}\right) > (0,0)$, which implies

$$\text{Sign}\left(\sigma_k(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(k)\right) = \text{Sign}\left(\sigma_s(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(s)\right). \tag{6}$$

We claim that

$$\sigma_k(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(k) \neq 0. \tag{7}$$

Suppose, on the contrary, $\sigma_k(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(k) = 0$. Then, by equality (6), $\sigma_s(N,u_N) - r^{\{k,s\}}_{M,x}(u_N)(s) = 0$. Consequently,

$$\begin{aligned}
\sigma_k(N,u_N) = r^{\{k,s\}}_{M,x}(u_N)(k) = \max\{0, 0 - \sigma_l(N,u_N)\} &\geq 0, \\
\sigma_s(N,u_N) = r^{\{k,s\}}_{M,x}(u_N)(s) = \max\{0, 0 - \sigma_l(N,u_N)\} &\geq 0,
\end{aligned} \tag{8}$$

being $l \in N \setminus \{k,s\}$. Thus,

$$\sigma_k(N,u_N) = \sigma_s(N,u_N) \geq 0. \tag{9}$$

Now consider the max reduced game $\left(\{k,l\}, r_{M,x}^{\{k,l\}}(u_N)\right)$ relative to $\{k,l\}$ at $x = \sigma(N, u_N)$. From (9) it follows that

$$r_{M,x}^{\{k,l\}}(u_N)(k) = r_{M,x}^{\{k,l\}}(u_N)(l) = 0 \text{ and } r_{M,x}^{\{k,l\}}(u_N)(kl) = 1 - \sigma_s(N, u_N).$$

By **MC** and, for two-person games, **SI** and **SRAM**, there is a strict monotone path $f \in \mathcal{F}_{mon}$ such that,

$$\begin{aligned} \sigma_{|\{k,l\}}(N, u_N) &= \sigma\left(\{k,l\}, r_{M,x}^{\{k,l\}}(u_N)\right) \\ &= \sigma(\{k,l\}, \mathbf{0}) + f(\{k,l\}, 1 - \sigma_s(N, u_N)) \qquad (10) \\ &= f(\{k,l\}, 1 - \sigma_s(N, u_N)). \end{aligned}$$

Since $f \in \mathcal{F}_{mon}$ and, by (9), $\sigma_k(N, u_N) \geq 0$, we have that $1 - \sigma_s(N, u_N) \geq 0$. Consequently, $\sigma_l(N, u_N) = f_l(\{k,l\}, 1 - \sigma_s(N, u_N)) \geq 0$. By expression (8), this means that $\sigma_k(N, u_N) = \sigma_s(N, u_N) = 0$ and by **E** (see Lemma 2 and Remark 3), $\sigma_l(N, u_N) = 1$. Then, $\sigma_l(N, u_N) - r_{M,x}^{\{k,l\}}(u_N)(l) = 1 - 0 = 1$. Since expression (6) holds for any pair of agents, we have that $\sigma_k(N, u_N) - r_{M,x}^{\{k,l\}}(u_N)(k) > 0$, in contradiction with $\sigma_k(N, u_N) = 0$. This prove the claim, that is, inequality (7). But then, from (5) it follows (4). □

To prove Theorems 3 and 4, previously we need to connect *standardness*, the *dummy player property*, *individual rationality* and *equal aggregate monotonicity*.

Lemma 6. *Let σ be a single-valued solution on Γ. Then, the following statements are equivalent:*

(i) σ satisfies dummy player property and equal aggregate monotonicity for two-person games.

(ii) σ satisfies standardness.

(iii) σ satisfies individual rationality and equal aggregate monotonicity for two-person games.

Proof To check the implication $(i) \to (ii)$, let $N = \{i,j\} \in \mathcal{N}$ and $(N, v) \in \Gamma$. If $v(N) = v(i) + v(j)$, then by **DP** (for two-person games) it follows directly **ST**. If $v(N) \neq v(i) + v(j)$, consider the associated game (N, v') defined as follows: $v'(k) = v(k)$ for all $k \in N$, and $v'(N) = v(i) + v(j)$. By **EAM** and **DP** (for two-person games) we obtain $\sigma_i(N, v) = \sigma_i(N, v') + \frac{1}{2}(v(N) - v'(N)) = v(i) + \frac{1}{2}(v(N) - v(i) - v(j))$. In a similar way, $\sigma_j(N, v) = v(j) + \frac{1}{2}(v(N) - v(i) - v(j))$. This prove **ST**.

We omit the proofs of the straightforward implications $(ii) \to (iii)$ and $(iii) \to (i)$. $\square$

Now, we have all the tools to prove our characterization results.

Proof (**Theorem 1**) Let $w \in \mathbb{R}^{\mathbb{N}}_{++}$ be a list of positive weights. Clearly, ES^w satisfies w-**P**. Moreover, Calleja and Llerena (2016) show that it also satisfies **PC**. To prove uniqueness, suppose there is another single-valued solution σ on Γ satisfying these two properties. By Lemma 1, σ satisfies **E**. Let (N, v) be a game. If $|N| = 1$, by **E** we have $\sigma(N, v) = ES^w(N, v)$. If $|N| = 2$, by w-**P** we have $\sigma(N, v) = ES^w(N, v)$. Finally, if $|N| \geq 3$, fix $i \in N$ and take an arbitrary $j \in N \backslash \{i\}$. Let $N' = \{i, j\} \subset N$, then,

$$\begin{aligned}
\sigma_i(N, v) &= \sigma_i(N', r^{N'}_{P,x}(v)) \\
&= v(i) + \frac{w_i}{w_i + w_j} \left(r^{N'}_{P,x}(v)(N') - v(i) - v(j) \right) \\
&= v(i) + \frac{w_i}{w_i + w_j} \left(\sigma_i(N, v) + \sigma_j(N, v) - v(i) - v(j) \right),
\end{aligned}$$

where the first equality follows by **PC**, the second one by w-**P** (w.r.t. $w \in \mathbb{R}^{\mathbb{N}}_{++}$) and the definition of projected reduced game, and the last one by **E**. Reordering terms, we obtain

$$\begin{aligned}
\sigma_i(N, v) \left(1 - \frac{w_i}{w_i + w_j} \right) &= v(i) + \frac{w_i}{w_i + w_j} \left(\sigma_j(N, v) - v(i) - v(j) \right) \\
&= v(i) \left(1 - \frac{w_i}{w_i + w_j} \right) + \frac{w_i}{w_i + w_j} \left(\sigma_j(N, v) - v(j) \right),
\end{aligned}$$

or, equivalently,

$$\left(\sigma_i(N, v) - v(i) \right) w_j = \left(\sigma_j(N, v) - v(j) \right) w_i.$$

Note that this equality holds for all $j \in N \backslash \{i\}$. Adding up,

$$\left(\sigma_i(N, v) - v(i) \right) \sum_{j \in N \backslash \{i\}} w_j = w_i \sum_{j \in N \backslash \{i\}} \left(\sigma_j(N, v) - v(j) \right),$$

and summing $\left(\sigma_i(N, v) - v(i) \right) w_i$ to both sides of the equality we obtain,

$$\begin{aligned}
\left(\sigma_i(N, v) - v(i) \right) \sum_{j \in N} w_j &= w_i \sum_{j \in N} \left(\sigma_j(N, v) - v(j) \right) \\
&= w_i \left(v(N) - \sum_{j \in N} v(j) \right),
\end{aligned}$$

where the last equality follows from **E**. Hence,

$$\sigma_i(N,v) = v(i) + \frac{w_i}{\sum_{j\in N} w_j}\left(v(N) - \sum_{i\in N} v(i)\right) = ES_i^w(N,v).$$

□

Proof (**Theorem 2**)

(i) As we have seen in Theorem 1, for any list of positive weights w, ES^w satisfies **PC**. It is no difficult to see that it also satisfies, for two-person games, **SI** and **SRAM** (w.r.t.f^w as defined in Subsection 3.2). To show uniqueness, suppose there is a another single-valued solution σ satisfying the above three properties. By Lemma 2, σ satisfies **E**. We claim that σ satisfies $\bar{w}-$**P** w.r.t. a list of positive weights $\bar{w} \in \mathbb{R}^{\mathbb{N}}_{++}$ as defined in Hart and Mas-Colell (1989). Fix a player $\mathbf{l} \in \mathbb{N}$ and define

$$\bar{w}_k = \begin{cases} 1 & \text{if } k = \mathbf{l} \\ \frac{\sigma_k(\{k,\mathbf{l}\},u_{\{k,\mathbf{l}\}})}{\sigma_{\mathbf{l}}(\{k,\mathbf{l}\},u_{\{k,\mathbf{l}\}})} & \text{otherwise} \end{cases}$$

By Lemma 3 (1), $\bar{w}$ is well defined.

Let (N,v) be a game. If $N = \{i\}$, by **E** we have that $\sigma(\{i\},v) = v(i) + \frac{\bar{w}_i}{\bar{w}_i}(v(i) - v(i)) = ES^{\bar{w}}(\{i\},v)$. If $|N| = 2$ we distinguish two cases:

1. **Case 1**: $N = \{\mathbf{l}, i\}$.
   Let us denote $\alpha = v(N) - v(\mathbf{l}) - v(i)$.
   If $\alpha = 0$, then by **SI** (for two-person games) (see expression (1) in Lemma 2) we have $\sigma(N,v) = (v(\mathbf{l}), v(i)) = ES^{\bar{w}}(N,v)$.
   If $\alpha > 0$, then by **SI** (for two-person games) and **E**, for all $k \in N$, we have

$$\begin{aligned}
\sigma_k(N,v) &= \sigma_k(N, \alpha\cdot u_N + (v(\mathbf{l}), v(i)) \\
&= \alpha\cdot\sigma_k(N,u_N) + v(k) \\
&= \alpha\cdot\frac{\sigma_k(N,u_N)}{\sigma_{\mathbf{l}}(N,u_N) + \sigma_i(N,u_N)} + v(k) \\
&= \alpha\cdot\frac{\frac{\sigma_k(N,u_N)}{\sigma_{\mathbf{l}}(N,u_N)}}{1 + \frac{\sigma_i(N,u_N)}{\sigma_{\mathbf{l}}(N,u_N)}} + v(k) \\
&= \alpha\cdot\frac{\bar{w}_k}{\bar{w}_{\mathbf{l}} + \bar{w}_i} + v(k) \\
&= ES_k^{\bar{w}}(N,v)
\end{aligned}$$

   where the second and the third equalities follow by **SI** (for two-person games) and **E**, respectively.

If $\alpha < 0$, notice first that $v = (-\alpha)\cdot(-u_N)+(v(\mathbf{l}), v(i))$. By **SI** (for two-person games) and Lemma 3 (2), $\sigma(N, v) = \alpha \cdot \sigma(N, u_N) + (v(\mathbf{l}), v(i))$. Now, following the reasoning above we obtain $\sigma_k(N, v) = ES_k^{\bar{w}}(N, v)$, for all $k \in N$.

2. **Case 2**: $N = \{i, j\}$ and $\mathbf{l} \notin N$.
   By the definition of $\bar{w}$, Lemma 4 and **E**, it follows that

$$\begin{aligned}
\frac{\bar{w}_i}{\bar{w}_i+\bar{w}_j} &= \frac{1}{1+\frac{\bar{w}_j}{\bar{w}_i}} \\
&= \frac{1}{1+\frac{\sigma_j\left(\{j,\mathbf{l}\},u_{\{j,\mathbf{l}\}}\right)}{\sigma_\mathbf{l}\left(\{j,\mathbf{l}\},u_{\{j,\mathbf{l}\}}\right)}\cdot\frac{\sigma_\mathbf{l}\left(\{i,\mathbf{l}\},u_{\{i,\mathbf{l}\}}\right)}{\sigma_i\left(\{i,\mathbf{l}\},u_{\{i,\mathbf{l}\}}\right)}} \\
&= \frac{1}{1+\frac{\sigma_j\left(\{i,j,\mathbf{l}\},u_{\{i,j,\mathbf{l}\}}\right)}{\sigma_\mathbf{l}\left(\{i,j,\mathbf{l}\},u_{\{i,j,\mathbf{l}\}}\right)}\cdot\frac{\sigma_\mathbf{l}\left(\{i,j,\mathbf{l}\},u_{\{i,j,\mathbf{l}\}}\right)}{\sigma_i\left(\{i,j,\mathbf{l}\},u_{\{i,j,\mathbf{l}\}}\right)}} \\
&= \frac{1}{1+\frac{\sigma_j\left(\{i,j,\mathbf{l}\},u_{\{i,j,\mathbf{l}\}}\right)}{\sigma_i\left(\{i,j,\mathbf{l}\},u_{\{i,j,\mathbf{l}\}}\right)}} \\
&= \frac{1}{1+\frac{\sigma_j\left(\{i,j\},u_{\{i,j\}}\right)}{\sigma_i\left(\{i,j\},u_{\{i,j\}}\right)}} \\
&= \frac{\sigma_i\left(\{i,j\},u_{\{i,j\}}\right)}{\sigma_i\left(\{i,j\},u_{\{i,j\}}\right)+\sigma_j\left(\{i,j\},u_{\{i,j\}}\right)} \\
&= \sigma_i\left(\{i,j\}, u_{\{i,j\}}\right).
\end{aligned} \tag{11}$$

   Similarly,

$$\frac{\bar{w}_j}{\bar{w}_i+\bar{w}_j} = \sigma_j(\{i,j\}, u_{\{i,j\}}). \tag{12}$$

   Let us denote $\alpha = v(N) - v(i) - v(j)$. If $\alpha > 0$, then **SI** (for two-person games) together with (11) imply

$$\begin{aligned}
\sigma_i(\{i,j\}, v) &= \sigma_i(\{i,j\}, (v(ij) - v(i) - v(j)) \cdot u_{\{i,j\}} + (v(i), v(j))) \\
&= (v(ij) - v(i) - v(j)) \cdot \sigma_i(\{i,j\}, u_{\{i,j\}}) + v(i) \\
&= (v(ij) - v(i) - v(j)) \cdot \frac{\bar{w}_i}{\bar{w}_i + \bar{w}_j} + v(i) \\
&= ES_i^{\bar{w}}(\{i,j\}, v)
\end{aligned}$$

   In a similar way, **SI** (for two-person games) together with (12) imply

$$\sigma_j(\{i,j\}, v) = ES_j^{\bar{w}}(\{i,j\}, v).$$

   If $\alpha \leq 0$, the same arguments used in Case 1 apply in this case. Consequently, σ satisfies $\bar{w}-$**P**. Finally, by Theorem 1 we conclude that $\sigma = ES^{\bar{w}}$.

(ii) As we have commented before, Lemmas 2, 3 and 4 hold by replacing *projected consistency* by *self consistency*. This fact, together with Theorem 5.7 in Hart and Mas-Colell (1989), conclude the proof.

□

Proof (**Theorem 6**) Suppose, on the contrary, that there exists a single-valued solution σ satisfying **MC**, **SI** and **SRAM** on Γ. Since Lemma 2 holds for **MC** (see Remark 3), from Lemma 3 and Lemma 5, it is not difficult to check, by following the same arguments as in the proof of Theorem 2 (i), that σ satisfies $w-$**P** (w.r.t. a list of positive weights as defined in the proof of Theorem 2 (i)). Consequently, σ satisfies $2-$**WS**. Since **SI** and **MC** jointly imply **E** (see Remark 3), and **SI** implies *zero-independence*, from Theorem 1 in Hokari (2005) it follows that σ coincides with ν_*. But ν_* fails to satisfies *aggregate monotonicity* (see, for instance, Hokari, 2000), contradicting **SRAM**. □

Proof (**Theorem 7**) Suppose, on the contrary, that there exists a single-valued solution σ satisfying **EAM**, **DP** and **MC** on Γ. Let (N, v) be a game with set of player $N = \{1, 2, 3\}$ and characteristic function $v(1) = 0$ and $v(S) = 1$ otherwise. Since player 1 is dummy in (N, v), by **DP** $\sigma_1(N, v) = 0$. Now consider the associated root-game (N, v_r).[11] Let $\left(\{2, 3\}, r^{\{2,3\}}_{M,x}(v_r)\right)$ be the max reduced game of (N, v_r) relative to $\{2, 3\}$ at $x = \sigma(N, v_r)$. By **DP** and **EAM**, $\sigma_1(N, v_r) = \sigma_1(N, v) + \bar{f}_1(\{1, 2, 3\}, 1) = \frac{1}{3}$. Thus, $r^{\{2,3\}}_{M,x}(v_r)(2) = 1$, $r^{\{2,3\}}_{M,x}(v_r)(3) = 1$ and $r^{\{2,3\}}_{M,x}(v_r)(23) = 2 - \frac{1}{3} = \frac{5}{3}$. Since players 2 and 3 are dummy players in the associated root-game $\left(\{2, 3\}, \left(r^{\{2,3\}}_{M,x}(v_r)\right)_r\right)$, by **DP** $\sigma\left(\{2, 3\}, \left(r^{\{2,3\}}_{M,x}(v_r)\right)_r\right) = (1, 1)$. By **EAM** $\sigma\left(\{2, 3\}, r^{\{2,3\}}_{M,x}(v_r)\right) = (1, 1) + \bar{f}(\{2, 3\}, \frac{-1}{3}) = (1, 1) + \left(\frac{-1}{6}, \frac{-1}{6}\right) = \left(\frac{5}{6}, \frac{5}{6}\right)$. Consequently, by **MC**

$$\sigma(N, v_r) = \left(\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right). \tag{13}$$

Now consider the max reduced game $\left(\{1, 2\}, r^{\{1,2\}}_{M,x}(v_r)\right)$ relative to $\{1, 2\}$ at $x = \sigma(N, v_r)$: $r^{\{1,2\}}_{M,x}(v_r)(1) = \frac{1}{6}$, $r^{\{1,2\}}_{M,x}(v_r)(2) = 1$ and $r^{\{1,2\}}_{M,x}(v_r)(12) = 2 - \frac{5}{6} = \frac{7}{6}$. Since players 1 and 2 are dummy players in the max reduced game $\left(\{1, 2\}, r^{\{1,2\}}_{M,x}(v_r)\right)$, by **DP** $\sigma\left(\left(\{1, 2\}, r^{\{1,2\}}_{M,x}(v_r)\right)\right) = \left(\frac{1}{6}, 1\right)$. Consequently, by **MC** $\sigma_{|\{1,2\}}(N, v_r) = \left(\frac{1}{6}, 1\right)$, in contradiction with (13). □

Appendix B

This appendix contains the independence of the properties used in the characterization results.

1. To prove that properties in Theorem 1 are independent, notice that, for a given $w \in \mathbb{R}^{\mathbb{N}}_{++}$, Sh^w satisfies *w-proportionality* but not *projected consistency.*

[11] Let $N \in \mathcal{N}$ and $(N, v) \in \Gamma$. The **root-game** associated to (N, v), denoted by (N, v_r), is the balanced game with smallest worth for the grand coalition N such that $v_r(S) = v(S)$ for all $S \subset N$.

Let $w' \in \mathbb{R}^{\mathbb{N}}_{++}$ be a different list of positive weights. Then, $ES^{w'}$ satisfies *projected consistency* but not *w-proportionality* (w.r.t. w). The independence of the properties in Corollary 1 follows similar arguments.

2. The see that the properties in Theorem 2 and Corollary 2 independent, let us first introduce the following single-valued solutions:

   Let π be a permutation on $\mathbb{N}$, the $f^{\pi}-$**surplus division**, denoted by $ES^{f^{\pi}}$, is defined as follows: for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in N$,

$$ES_i^{f^{\pi}}(N,v) = v(i) + f_i^{\pi}\left(N, v(N) - \sum_{i\in N} v(i)\right),$$

   being f^{π} as defined in Subsection 3.2.

   The **equal divison solution**, denoted by ED, is defined as follows: for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in N$,

$$ED_i(N,v) = \frac{v(N)}{n}.$$

   Let π be a permutation on $\mathbb{N}$, the **marginal contribution solution** relative to π, denoted by mc^{π}, is defined as follows: for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in N$

$$mc_i^{\pi} := v\left(\{j \in N \mid \pi(j) \leq \pi(i)\}\right) - v\left(\{j \in N \mid \pi(j) < \pi(i)\}\right).$$

   It is not difficult to check that $ES^{f^{\pi}}$ satisfies *scale invariance* and *projected consistency* but not *strong regular aggregate monotonicity. Sh* satisfies *scale invariance* and *equal aggregate monotonicity* (and thus *strong regular aggregate monotonicity*) but not *projected consistency. ED* satisfies *equal aggregate monotonicity*, *projected consistency* and *self consistency* but not *scale invariance* for two-person games. On the other hand, mc^{π} satisfies *scale invariance* and *self consistency* but not *strong aggregate monotonicity* for two-person games. ES satisfies *scale invariance* and *equal aggregate monotonicity* but not *self consistency.*

3. To see that the properties in Theorem 3 are independent, let us first introduce the following single-valued solution $\boldsymbol{\rho}$: for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in N$,

$$\boldsymbol{\rho}_i(N,v) = x_i + \frac{1}{n}\left(v(N) - x(N)\right),$$

   where $x_i = v(i)$ if $i \in PD(N,v)$, and $x_i = \frac{v(N \setminus PD(N,v))}{|v(N \setminus PD(N,v)|}$ otherwise, being $PD(N,v) = \{i \in N \mid v(S \cup \{i\}) - v(S) = v(i) \; \forall\, S \subset N \setminus \{i\}\}.$

The single-valued solution $\boldsymbol{\rho}$ satisfies the *dummy player* property and *equal aggregate monotonicity* but neither *self consistency* nor *projected consistency.* ED satisfies *equal aggregate monotonicity*, *self consistency* and *projected consistency* but not the *dummy player* property for two-person games. $ES^{f^{\pi}}$ satisfies *projected consistency* and the *dummy player* property for two-person games but not *equal aggregate monotonicity* for two-person games. The marginal contribution solution mc^{π} satisfies *self consistency* and the *dummy player* property, but not *equal aggregate monotonicity* for two-player games.

4. To prove that the properties in Theorem 4 are independent, notice that for a suitable list of positive weights w, Sh^w satisfies *self consistency*, *individual rationality* for two-person games, but not *equal aggregate monotonicity.* ES^w satisfies satisfies *projected consistency*, *individual rationality*, but not *equal aggregate monotonicity* for two-person games. ED satisfies *self consistency*, *projected consistency*, *equal aggregate monotonicity* but not *individual rationality* for two-person games. ES satisfies *projected consistency*, *individual rationality*, *equal aggregate monotonicity* but not *self consistency.* Sh satisfies *self consistency*, *individual rationality* for two-person game, *equal aggregate monotonicity* but not *projected consistency.*

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