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Compromise solutions for bankruptcy situations: a note

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Abstract

Although classic bankruptcy problems take into account a single claims vector, Pulido et al. (2008) show that there are real bankruptcy situations where agents face more than one reference vector. In particular, they consider the claims and an additional reference vector. To analyze these situations, they propose the extreme and the diagonal approaches. Nonetheless, the former approach depends on the order of the vectors: if we interchange the claims and the reference vectors, the result changes. Moreover their study is limited to the case in which the reference vector is lower than the claims vector. In the present note, we propose an extension that solves these shortcomings by introducing the idea of impartiality.

Keywords: bankruptcy problems; reference point; compromise solution; impartiality

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1. Introduction

In a bankruptcy situation, a given amount of money (the estate) has to be allocated among a set of agents, each of whom has a claim on this estate. The total amount claimed exceeds the estate available, so not all the claims can be fully honored. This typical bankruptcy problem, introduced by O'Neill (1982), was enriched with the contributions of Pulido et al. (2002, 2008) who show that other references, in addition to the claims, might be relevant to allocate the estate in some contexts.

The present note attempts to complete their work, by extending the set of situations in which it can be applied. By considering two independent and general reference points, we drop out the condition that the reference point is dominated by the claims. Accordingly, we modify the compromise solutions introduced by Pulido et al. (2008) in order to make the result independent on which reference point is taken as claims vector (we call this property impartiality).

The paper is organized as follows. Next section introduces the model and main notation. Section 3 extends the extreme and diagonal compromise approaches. Finally, the paper finishes with some comments about the cooperative game associated to these approaches.

2. The Model and notation

A bankruptcy problem (O'Neill, 1982) is a triple $(N, E, \mathbf{z})$, where N is the finite set of agents, $N = \{1, 2, \ldots, n\}$, $E \geq 0$ is the estate to be divided among them, and $\mathbf{z} \in \mathbb{R}^n_+$ is the vector of claims such that $Z = \sum_{i \in N} z_i \geq E$. A rule is a function φ that assigns to every bankruptcy situation $(N, E, \mathbf{z})$ a vector $\varphi(N, E, \mathbf{z}) \in \mathbb{R}^n$ such that $0 \leq \varphi_i(N, E, \mathbf{z}) \leq z_i$ for all $i \in N$ (non-negativity and boundedness), and $\sum_{i \in N} \varphi_i(N, E, \mathbf{z}) = E$ (efficiency).

In Pulido et al. (2002, 2008) the bankruptcy problem has been extended by adding an additional point $\mathbf{r} \in \mathbb{R}^n_+$, called the reference point, such that, for all $i \in N$, $r_i \leq z_i$. Then, a bankruptcy problem with references is a 4-tuple $(N, E, \mathbf{r}, \mathbf{z})$, where $(N, E, \mathbf{z})$ is a bankruptcy problem and $\mathbf{r} \leq \mathbf{z}$. Moreover, they distinguish two situations: (CREO) $R = \sum_{i \in N} r_i \geq E$; and (CERO) $R = \sum_{i \in N} r_i < E$.

We extend this model by considering that the claims vector and the reference vector are not related. So, we may have $r_i < z_i$ for some agents, and $r_i > z_i$ for some other agents.

Definition 1. *A bankruptcy problem with two reference vectors* $\mathbf{x}$, $\mathbf{y}$ *is a 4-tuple* $(N, E, \mathbf{x}, \mathbf{y})$, *where* $\mathbf{x} \in \mathbb{R}^n_+$ *and* $\mathbf{y} \in \mathbb{R}^n_+$ *are such that:*

$$X = \sum_{i \in N} x_i > E, \ \textit{or} \ Y = \sum_{i \in N} y_i > E.$$

Note that we do not assume any condition about the two reference vectors, $\mathbf{x}$ and $\mathbf{y}$, so both the (CERO) and (CREO) cases are included. We extend these situations by allowing that no vector dominates the other. The only condition that we ask for ($X > E$ or $Y > E$) is just to ensure that one of the problems (either $(N, E, \mathbf{x})$ or $(N, E, \mathbf{y})$) is a bankruptcy problem.

3. Compromise solutions

Note that given a bankruptcy problem with two references, $(N, E, \mathbf{x}, \mathbf{y})$, if we assume $x_i \leq y_i$, for all $i \in N$, and $Y = \sum_{i \in N} y_i \geq E$, it corresponds to the case analyzed in Pulido et al. (2008). In this context they define two solutions: the extreme compromise and the diagonal compromise solutions. In order to define their solutions, they consider a given rule φ. As they mention, φ "reflects some benchmark rule for evaluating claims and can be interpreted, e.g., as the method that was used on a previous occasion to solve a similar bankruptcy problem" (Pulido et al., 2008).

3.1. ***The impartial extreme solution***

Accordingly to some benchmark rule φ, Pulido et al. (2008) approach determines for all agent the combination of references and claims that leads to the lowest (l^φ) and highest (L^φ) allocations, defined by[1].

$$l_i^\varphi(N, E, \mathbf{x}, \mathbf{y}) = \begin{cases} \varphi_i(N, E, (x_i, \mathbf{y}_{-i})) & \text{if} \quad x_i + \sum_{j \neq i} y_j \geq E \\ E - \sum_{j \neq i} y_j & \text{if} \quad x_i + \sum_{j \neq i} y_j < E \end{cases}$$

$$L_i^\varphi(N, E, \mathbf{x}, \mathbf{y}) = \begin{cases} \varphi_i(N, E, (y_i, \mathbf{x}_{-i})) & \text{if} \quad y_i + \sum_{j \neq i} x_j \geq E \\ y_i & \text{if} \quad y_i + \sum_{j \neq i} x_j < E \end{cases}$$

Note that whenever for all $i \in N$ $x_i \leq y_i$ and $Y = \sum_{i \in N} y_i \geq E$, then $l_i^\varphi(N, E, \mathbf{x}, \mathbf{y}) \leq L_i^\varphi(N, E, \mathbf{x}, \mathbf{y})$.

Then, the *compromise solution* γ^φ is defined as the unique efficient convex combination of these two vectors (called extreme vectors),

[1] In the CREO case, only the first part of the definition applies.

$$\gamma^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = \alpha l^{\varphi}(N,E,\mathbf{x},\mathbf{y}) + (1-\alpha)\, L^{\varphi}(N,E,\mathbf{x},\mathbf{y}),$$

where $\alpha \in [0,1]$ is such that $\sum_{i\in N} \gamma_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = E$.

If we try to extend this solution for general reference vectors $\mathbf{x}$ and $\mathbf{y}$, in which none of them dominates the other, the first decision is to choose which one plays the role of the claims vector and which one is the reference vector. The following example illustrates that this question is essential in order to apply the extreme compromise solution.

Example 1. *Consider the bankruptcy problem with references* $(N,E,\mathbf{x},\mathbf{y})$ *defined by* $N = \{1,2,3\}$, $E = 100$ *and the reference vectors* $\mathbf{x} = (20,25,40)$ *and* $\mathbf{y} = (35,50,20)$. *Consider that the benchmark bankruptcy rule is the proportional rule,* $\varphi = Pr$. *Following the Pulido et al. (2008) extreme approach, if we make* $\mathbf{x}$ *the reference vector and* $\mathbf{y}$ *the claims vector, the lower and upper extreme vectors are, respectively,*

$$l^{\varphi} = (30,45,32) \text{ and } L^{\varphi} = (35,45.45,20)\,.$$

The associated extreme compromise solution is $\gamma^{\varphi} = (35.35,45.49,19.16)$ *obtained with a value of* $\alpha = -0.0694$.

However, if we make $\mathbf{x}$ *the claims vector and* $\mathbf{y}$ *the reference vector, the new lower and upper extreme vectors are, respectively,*

$$l^{\varphi} = (35,45.45,55) \text{ and } L^{\varphi} = (20,25,32)\,.$$

The associated extreme compromise solution is $\gamma^{\varphi} = (25.90,33.05,41.05)$ *obtained with a value of* $\alpha = 0.3935$.

The above example shows that when we invert the order of the vectors, the lower bound vector becomes an upper bound and vice versa. Henceforth, the extreme compromise solution depends on which we consider to be the reference and the claims vectors. In each case, some individuals in the population N are benefited while others are penalized. Moreover, there are situations in which none of the extreme vector strictly dominates the other. It is also clear from the example that each ordering has associated a different extreme compromise solution. In addition, the value of α may fail to be in the interval $[0,1]$.

We extend the extreme approach in the sense of treating both vectors symmetrically (no proposal is more important than the other), either because we do not want to discriminate one proposal over the other, or because we do not have sufficient information to take such position (see, for instance, Marco-Gil et al. (1995) and Gadea-Blanco et al. (2010)).

IMPARTIALITY: A solution χ^{φ} for bankruptcy problems with two references is said to be (reference) ***impartial*** if for any bankruptcy problem with two reference vectors $(N, E, \mathbf{x},\ \mathbf{y})$, then $\chi^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \chi^{\varphi}(N, E, \mathbf{y}, \mathbf{x})$.

Definition 2. *Let $(N, E, \mathbf{x}, \mathbf{y})$ be a bankruptcy problem with two reference vectors and φ a benchmark rule. The impartial extreme vectors are defined by*

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \min\{l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}), l_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x})\},$$
$$M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \max\{L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}), L_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x})\}.$$

In the following result we show that the two impartial extreme vectors in our general setting satisfy a condition needed to properly define the extreme compromise solution; namely, one of them is below the other, $\mathbf{m}^{\varphi} \leq \mathbf{M}^{\varphi}$.

Lemma 1. *Let $(N, E, \mathbf{x}, \mathbf{y})$ be a bankruptcy problem with two reference vectors and φ a benchmark rule. Then,*

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \ \leq \ M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \qquad \forall i \in N.$$

Proof. We consider four cases.
[CASE 1] Suppose $x_i + \sum_{j \neq i} y_j \geq E$ and $y_i + \sum_{j \neq i} x_j \geq E$. Then by definition of extreme vectors,

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \min\{\varphi_i(N, E, (x_i, \mathbf{y}_{-i})), \varphi_i(N, E, (y_i, \mathbf{x}_{-i}))\}, \text{ and}$$
$$M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \max\{\varphi_i(N, E, (x_i, \mathbf{y}_{-i})), \varphi_i(N, E, (y_i, \mathbf{x}_{-i}))\},$$

so the result obviously holds.

[CASE 2] Now consider $x_i + \sum_{j \neq i} y_j \geq E$ and $y_i + \sum_{j \neq i} x_j < E$. Therefore,

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \min\{\varphi_i(N, E, (x_i, \mathbf{y}_{-i})), E - \textstyle\sum_{j \neq i} y_j\}, \text{ and}$$
$$M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \max\{\varphi_i(N, E, (x_i, \mathbf{y}_{-i})), y_i\}.$$

Then,

$$m_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) \leq \varphi_i(N,E,(x_i,\mathbf{y}_{-i})) \leq M_i^{\varphi}(N,E,\mathbf{x},\mathbf{y})$$

and the result holds.

[CASE 3] If $x_i + \sum_{j\neq i} y_j < E$ and $y_i + \sum_{j\neq i} x_j \geq E$ we reason as in the previous case.

[CASE 4] Finally, consider $x_i + \sum_{j\neq i} y_j < E$ and $y_i + \sum_{j\neq i} x_j < E$. Then,

$$m_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = \min\{E - \textstyle\sum_{j\neq i} x_j, E - \sum_{j\neq i} y_j\}, \text{ and}$$
$$M_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = \max\{x_i, y_i\}.$$

Let us suppose, without losing generality, that $X = \sum_{i\in N} x_i > E$. Therefore $x_i > E - \sum_{j\neq i} x_j$, which implies $m_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) \leq M_i^{\varphi}(N,E,\mathbf{x},\mathbf{y})$. ■

From the impartial extreme vectors we define the compromise solution as the efficient point in the line joining these vectors.

Definition 3. *The* ***impartial extreme compromise solution*** ψ^{φ} *is defined by*

$$\psi_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = \alpha m_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) + (1-\alpha)M_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) \text{ for all } i \in N,$$

where $\alpha \in \mathbb{R}$ *is selected such that* $\sum_{i\in N} \psi_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = E$.

Example 2. *With the data in Example 1 we obtain:*

$$m^{\varphi} = (30,45,32) \qquad M^{\varphi} = (35,45.45,32)$$
$$\psi^{\varphi} = (23.58,44.42,32),$$

with a value of $\alpha = 2.2833$.

The next result shows some consequences of our definition. In particular the impartial objective is achieved. The immediate proof is omitted.

Proposition 1. *Let* $(N,E,\mathbf{x},\mathbf{y})$ *be a bankruptcy problem with two reference vectors and* φ *a benchmark rule. Then,*

a) The impartial extreme compromise solution is impartial, that is,

$$\psi_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) = \psi_i^{\varphi}(N,E,\mathbf{y},\mathbf{x}).$$

b) $\psi_i^{\varphi}(N,E,\mathbf{x},\mathbf{y}) \geq 0$, *for all* $i \in N$.

c) $\alpha \in \mathbb{R}_+$.

3.2. ***The extension result***

We now prove that our solution concept is an extension of the one defined in Pulido et al. (2008). Under their assumptions both approaches coincide. As in that paper, complementary monotonicity is assumed; that is, if the claim of an individual $i \in N$ increases by a certain amount, then the associated benchmark rule cannot allocate to her less than before.

Proposition 2. *Let $(N, E, \mathbf{x}, \mathbf{y})$ be a bankruptcy problem with two reference vectors such that $x_i \leq y_i$ for all $i \in N$, and φ a benchmark rule satisfying complementary monotonicity. Then,*

a)

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \text{ and}$$
$$M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}).$$

b)

$$\sum_{i \in N} m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \leq E \leq \sum_{i \in N} M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}).$$

and $\alpha \in [0, 1]$.

c) $\psi_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \gamma_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y})$, for all $i \in N$.

Proof. Part a): As in the proof of Lemma 1 we distinguish four cases. Note that, in all of them we assume

$$x_i \leq y_i \text{ for all } i \in N \text{ and } \sum_{i=1}^{n} y_i \geq E.$$

[Case 1] Suppose $x_i + \sum_{j \neq i} y_j \geq E$ and $y_i + \sum_{j \neq i} x_j \geq E$. Then by definition of extreme vectors,

$$l_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \text{ and}$$

$$L_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y})$$

so the result follows from Lemma 1 in Pulido et al. (2008).

[Case 2] Now consider $x_i + \sum_{j\neq i} y_j \geq E$ and $y_i + \sum_{j\neq i} x_j < E$. Therefore,

$$l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \varphi_i(N, E, (x_i, \mathbf{y}_{-i})) \text{ and } \; l_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = E - \sum_{j\neq i} x_j$$

Then,

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}).$$

On the other hand,

$$L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = y_i \text{ and } \; L_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = \varphi_i(N, E, (y_i, \mathbf{x}_{-i}))$$

and the result holds.

[Case 3] If $x_i + \sum_{j\neq i} y_j < E$ and $y_i + \sum_{j\neq i} x_j \geq E$, then

$$l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = E - \textstyle\sum_{j\neq i} y_j \text{ and } \; l_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = \varphi_i(N, E, (y_i, \mathbf{x}_{-i}))$$

Then, as the second element is bigger than $E - \sum_{j\neq i} x_j$,

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}).$$

On the other hand,

$$L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \varphi_i(N, E, (y_i, \mathbf{x}_{-i})) \text{ and } \; L_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = x_i$$

and the result holds.

[Case 4] Finally, consider $x_i + \sum_{j\neq i} y_j < E$ and $y_i + \sum_{j\neq i} x_j < E$. Then,

$$l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = E - \textstyle\sum_{j\neq i} y_j \text{ and } \; l_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = x_i,$$

$$L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = y_i \text{ and } \; L_i^{\varphi}(N, E, \mathbf{y}, \mathbf{x}) = E - \sum_{j\neq i} x_j,$$

and the result holds.

Parts b) and c) are immediate from part a) and the results in Pulido et al. (2008). ■

In our general setting with arbitrary reference vectors in some cases the left-hand side inequality in Part b) of Proposition 2 is not guaranteed. The example 2 above shows this issue. If we add the elements in the vector m^{φ} we obtain $\sum_{i\in N} m_i^{\varphi} = 107$ which is above the estate value $E = 100$. The implication is that the value of α is outside the interval $[0,1]$. However, from the proof of the above result it is clear that whenever the reference vectors are in Case 1, then Proposition 2 is always true, even in the case that no vector dominates the other. Then, we have the following immediate consequence.

Corollary 1. *Let $(N, E, \mathbf{x}, \mathbf{y})$ be a bankruptcy problem with two reference vectors such that $x_i + \sum_{j\neq i} y_j \geq E$ and $y_i + \sum_{j\neq i} x_j \geq E$, for all $i \in N$, and φ a benchmark rule satisfying complementary monotonicity. Then,*

a)

$$m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = l_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \;\; \text{and}$$
$$M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = L_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}).$$

b)

$$\sum_{i\in N} m_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) \;\; \leq E \leq \;\; \sum_{i\in N} M_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}).$$

and $\alpha \in [0,1]$.

c) $\psi_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \gamma_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y})$, for all $i \in N$.

3.3. ***The extended diagonal solution***

The diagonal approach proposed by Pulido et al. (2008) does not change with the order of the reference vectors, since the parameter $\lambda \in [0,1]$ adjusts to make the compensation. Consequently, the obtained vectors are lower and upper bounds, respectively. However, when extended for general reference vectors it suffers from other problem. In order to see it consider the Pulido et al. (2008) definition. The *diagonal approach* is defined by a lower value,

$$\bar{l}_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \min_{\lambda\in[0,1]} h_i^{\varphi,\lambda}(N, E, \mathbf{x}, \mathbf{y}),$$

and a upper value,

$$\overline{L}_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \max_{\lambda\in[0,1]} h_i^{\varphi,\lambda}(N, E, \mathbf{x}, \mathbf{y}),$$

for all $i \in N$, where,

$$h_i^{\varphi,\lambda}(N, E, \mathbf{x}, \mathbf{y}) = \begin{cases} \varphi_i(N, E, \lambda\mathbf{x} + (1-\lambda)\,\mathbf{y}) & if \quad \lambda X + (1-\lambda)\,Y \geq E, \\ \lambda x_i + (1-\lambda)\,y_i + \varphi_i(\overline{E}^{\lambda}, \overline{\mathbf{d}}^{\lambda}) & if \quad \lambda X + (1-\lambda)\,Y < E, \end{cases}$$

with

$$\overline{E}^{\lambda} = E - (\lambda X + (1-\lambda) Y),$$

and

$$\overline{\mathbf{d}}^{\lambda} = \mathbf{y} - (\lambda \mathbf{x} + (1-\lambda) \mathbf{y}). \quad (1)$$

In our general setting, vectors can be arbitrary and $x_i \leq y_i$ is not assumed. Therefore, the residual vector $\overline{\mathbf{d}}^{\lambda}$ must be adjusted in order to not deliver negative values ($\overline{d}_i^{\lambda} = \lambda(y_i - x_i) < 0$ in the case $x_i > y_i$). Otherwise, we may obtain nonsense and counter-intuitive results. In expression (1) the first value of y_i is replaced by $\max\{x_i, y_i\}$.

Definition 4. *Let us consider a bankruptcy problem with two reference vectors* $(N, E, \mathbf{x}, \mathbf{y})$, *and a benchmark rule* φ. *The adjusted residual vectors are defined as*

$$\widehat{d}_i^{\lambda} = \max\{x_i, y_i\} - (\lambda x_i + (1-\lambda) y_i) > 0, \quad (2)$$

for all $i \in N$. *The* ***adjusted diagonal compromise solution*** *is*

$$\widehat{\psi}^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \alpha \widehat{l}^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) + (1-\alpha) \widehat{L}^{\varphi}(N, E, \mathbf{x}, \mathbf{y}),$$

where $\alpha \in [0,1]$ *is such that* $\sum_{i \in N} \widehat{\psi}_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = E$.

The properties of the solution stated in Lemma 1 of Pulido et al. (2008) hold true for the *adjusted diagonal compromise solution*. Note also that the diagonal solution is a particular case of the adjusted diagonal solution. If $x_i \leq y_i$ for all $i \in N$, then $\widehat{\psi}_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y}) = \overline{\gamma}_i^{\varphi}(N, E, \mathbf{x}, \mathbf{y})$.

The following example compares our solution with the diagonal compromise one.

Example 3. *With the data in Example 1, the diagonal compromise solution equals to*

$$\overline{\gamma}^{\varphi} = (32.29, 45.68, 22.02),$$

and the adjusted diagonal compromise solution equals to

$$\widehat{\psi}^{\varphi} = (29.48, 41, 29.52),$$

both obtained with a value of $\alpha = 0.5$.

Note that individual $i = 3$ diagonal allocation equals 22.02 which is too close to 20 and too far from 40, while the adjusted diagonal allocation equals 29.52 which is a better compromise between the reference claims 20 and 40. The same argument holds for $i = 1, 2$.

4. Final remarks

In Pulido et al. (2008) a cooperative analysis of their solutions has been carried out. Their main result shows that the compromise solutions (both the extreme and the diagonal ones) can be obtained as the τ value (see, Tijs (1981), Driessen and Tijs (1985)) of a bankruptcy cooperative game, by defining appropriate characteristic functions. An analogous study can be carried out in our general setting, just by introducing in the characteristic functions the same modifications done in both compromise solutions.

In the case of the extreme compromise solution, Pulido et al. (2008) define a characteristic function v^{φ} to introduce the so-called extreme game. This function depends on the subset of agents $S \subseteq N$ considered and the reference vectors $(\mathbf{x}, \mathbf{y})$, $v^{\varphi}(S, \mathbf{x}, \mathbf{y})$. We easily note that this function is not (reference) impartial. To solve this problem, we can define:

$$v^{\psi}(S, \mathbf{x}, \mathbf{y}) = \min\{v^{\varphi}(S, \mathbf{x}, \mathbf{y}), v^{\varphi}(S, \mathbf{y}, \mathbf{x})\},$$

that is, for all coalition S we consider the worst scenario over the two games $v^{\varphi}(S, \mathbf{x}, \mathbf{y})$ and $v^{\varphi}(S, \mathbf{y}, \mathbf{x})$.

Within this framework, we obtain (the proof runs parallel to the one in Pulido et al. (2008)) that the τ value of this game coincides with the impartial extreme compromise solution:

$$\psi^{\varphi} = \tau(v^{\psi}).$$

A similar reasoning can be made with respect to the adjusted diagonal compromise solution. The adjusted diagonal game follows closely the original diagonal game but with the elements in the residual vector (1) defined as in (2).

References

Driessen, T., Tijs, S., 1985. Tau value, core and semiconvex games. International Journal of Game Theory 14 (4), 229–248.

Gadea-Blanco, P., Jiménez-Gómez, J. M., Marco-Gil, M. C., 2010. Some game-theoretic grounds for meeting people half-way. IVIE. Working Papers. Serie AD 04.

Marco-Gil, M. C., Peris, J. E., Subiza, B., 1995. A mechanism for meta-bargaining problems. IVIE. Working Papers. Serie AD 98-17.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2 (4), 345–371.

Pulido, M., Borm, P., Hendrickx, R., Llorca, N., Sánchez-Soriano, J., 2002. Game theory techniques for university management: an extended bankruptcy model. Annals of Operations Research 109, 129–142.

Pulido, M., Borm, P., Hendrickx, R., Llorca, N., Sánchez-Soriano, J., 2008. Compromise solutions for bankruptcy situations with references. Annals of Operations Research 158 (1), 133–141.

Tijs, S., 1981. Bounds for the core and the τ-value, In *Game Theory and Mathematical Economics* (O. Moeschlin and D. Pallaschke, eds) North-Holland, Amsterdam. North-Holland, Amsterdam, pp. 123 – 132.