UNIVERSITAT
ROVIRA I VIRGILI
DEPARTAMENT D'ECONOMIA

CREIP
CENTRE DE RECERCA EN ECONOMIA INDUSTRIAL I ECONOMIA PÚBLICA

WORKING PAPERS

Col·lecció "DOCUMENTS DE TREBALL DEL DEPARTAMENT D'ECONOMIA - CREIP"

Axiomatization of the nucleolus of assignment games

Francesc Llerena
Marina Núñez
Carles Rafels

Document de treball n.06- 2014

DEPARTAMENT D'ECONOMIA – CREIP
Facultat d'Economia i Empresa

UNIVERSITAT
ROVIRA I VIRGILI
DEPARTAMENT D'ECONOMIA

(creip)
CENTRE DE RECERCA EN ECONOMIA
INDUSTRIAL I ECONOMIA PÚBLICA

Edita:

Departament d'Economia
www.fcee.urv.es/departaments/economia/public_html/index.html
Universitat Rovira i Virgili
Facultat d'Economia i Empresa
Avgda. de la Universitat, 1
43204 Reus
Tel.: +34 977 759 811
Fax: +34 977 300 661
Email: sde@urv.cat

CREIP
www.urv.cat/creip
Universitat Rovira i Virgili
Departament d'Economia
Avgda. de la Universitat, 1
43204 Reus
Tel.: +34 977 558 936
Email: creip@urv.cat

Adreçar comentaris al Departament d'Economia / CREIP

Dipòsit Legal: T - 566 - 2014

ISSN edició en paper: 1576 - 3382
ISSN edició electrònica: 1988 - 0820

DEPARTAMENT D'ECONOMIA – CREIP
Facultat d'Economia i Empresa

Axiomatization of the nucleolus of assignment games

Francesc Llerena[a], Marina Núñez[b], Carles Rafels[b]

[a]*Dep. de Gestió d'Empreses, Universitat Rovira i Virgili-CREIP,*
Av. de la Universitat, 1, 43204 Reus, Spain
[b]*Dep. de Matemàtica Econòmica, Financera i Actuarial*
Universitat de Barcelona.
Av. Diagonal 690, 08034 Barcelona, Spain

Abstract

On the domain of general assignment games (with possible reservation prices) the core is axiomatized as the unique solution satisfying two consistency principles: projection consistency and derived consistency. Also, an axiomatic characterization of the nucleolus is given as the unique solution that satisfies derived consistency and equal maximum complaint between groups. As a consequence, we obtain a geometric characterization of the nucleolus. Maschler et al. (1979) provide a geometrical characterization for the intersection of the kernel and the core of a coalitional game, showing that those allocations that lie in both sets are always the midpoint of certain bargaining range between each pair of players. In the case of the assignment game, this means that the kernel can be determined as those core allocations where the maximum amount, that can be transferred without getting outside the core, from one agent to his/her optimally matched partner equals the maximum amount that he/she can receive from this partner, also remaining inside the core. We now prove that the nucleolus of the assignment game can be characterized by requiring this bisection property be satisfied not only for optimally matched pairs but also for optimally matched coalitions.

Key words:
cooperative games, assignment game, core, nucleolus

1. Introduction

The assignment game is a coalitional game introduced by Shapley and Shubik (1972) to analyze a two-sided market situation. In this market there exists a finite set of sellers, each one of them with an indivisible object on sell, and a finite set of buyers willing to buy at most one object each. Objects are distinct and buyers may value them differently. From these valuations we obtain a non-negative matrix A that gives the profit a_{ij} that each buyer-seller pair (i, j) can attain if they interchange the object. The worth of each coalition is the total profit that can be obtained by optimally matching buyers and sellers in the coalition.

Cooperative game theory analyzes how the agents can share the profit of an optimal pairing, taking into account the worth of all possible coalitions. The most studied solution concept in

[1]The authors acknowledge the support from research grants ECO2008-02344/ECON (Ministerio de Ciencia e Innovación and FEDER), 2009SGR900 and 2009SGR960 (Generalitat de Catalunya).

this model has been the core, the set of efficient allocations that are coalitionally stable (each coalition gets at least its worth). Shapley and Shubik prove that the core of the assignment game is non-empty and coincides with the set of dual optimal solutions of the assignment optimization problem that solves the worth of the grand coalition. As a consequence, the core of the assignment game coincides with the set of competitive equilibria and it can be described just in terms of the assignment matrix, with no need of the coalitional worths.

Other solutions have been considered for the assignment game: the kernel or symmetrically pairwise bargained allocations (Rochford, 1984), the tau value (Núñez and Rafels, 2002), the Shapley value (Hoffmann and Sudhölter, 2007) and the von Neumann-Morgenstern stable sets (Núñez and Rafels, 2009). However, as far as we know, axiomatic characterizations of solutions in this framework have been focused on the core. A first axiomatization of the core of the assignment game is due to Sasaki (1995).

Since our purpose is also axiomatization and most of the known solutions are covariant with respect to strategic equivalence, it is desirable to work with a class of games that is closed under strategic equivalence. To this end we consider assignment games with reservation prices (when an agent is unmatched the profit she generates is her reservation price) and where the assignment matrix A is not constrained to be non-negative. In this more general framework Toda (2005) provides two axiomatizations of the core of the assignment game in terms of Pareto optimality, consistency, pairwise monotonicity and either individual monotonicity or population monotonicity. Here consistency refers to projection consistency and requires that, for each solution outcome, the same outcome should be recommended for each subgame that results when some agents leave with what they have received.

Davis and Maschler (1965) formulate a consistency axiom on the domain of general coalitional games, which is called max-consistency by Thomson (2003). It requires that for each solution outcome, if some agents leave the game and the remaining coalitions reevaluate their worth by considering the maximum profit they could obtain with some of the outside agents, after paying them according to the solution, the same outcome should be recommended to this max-reduced game. On the domain of coalitional games, solutions such as the core, the kernel and the nucleolus satisfy max-consistency (Peleg, 1986; Sobolev, 1975).

The max-reduced game of an assignment game, even if the leaving agents are paid according to a core element, need not be another assignment game since superadditivity may fail. However, its superadditive cover is what Owen (1992) defines as the derived assignment game and it turns out to be a general assignment game with reservation prices and arbitrary matrix. Thus, for general assignment games, max-consistency is equivalent to derived consistency. Toda (2003) gives characterizations of the core and the kernel on the domain of general assignment games, by means of derived consistency, that parallel the axiomatizations of Peleg (1986) for the core and the kernel of coalitional games.

In the present paper we also use derived consistency to give, on the domain of general assignment games, an axiomatic characterization of the nucleolus and a new axiomatization of the core. The core of the assignment game is axiomatized only by means of projection consistency and derived consistency. The nucleolus is the unique solution on the domain of general assignment games that satisfies derived consistency and equal maximum complaint between groups. This second axiom requieres that at each solution outcome, the maximum over individual excesses of buyers equals the maximum over individual excesses of sellers, and it is satisfied by other solutions like for instance the tau value.

As a by-product of the axiomatic characterization of the nucleolus we obtain a geometric characterization. Maschler et al. (1979) provide a geometrical characterization for the intersection of the kernel and the core of a coalitional game, showing that those allocations that lie in both sets are always the midpoint of certain bargaining range between each pair of players. In the case of the assignment game, this means that the kernel can be determined as those core allocations where the maximum amount, that can be transferred without getting outside the core, from one agent to his/her optimally matched partner equals the maximum amount that he/she can receive from this partner, also remaining inside the core. We now prove that the nucleolus of the assignment game can be characterized by requiring this bisection property be satisfied not only for optimally matched pairs but also for optimally matched coalitions.

Preliminaries on coalitional games and assignment games are in Section 2. Section 3 introduces the derived-game consistency and proves the axiomatic characterization of the core in terms of the two consistency properties. Section 4 is devoted to the axiomatic characterization of the nucleolus. Section 5 concludes with the geometric characterization of the nucleolus.

2. Preliminaries

Let N be an arbitrary non-empty finite set of players, and 2^N the set of all coalitions of N. A *transferable utility cooperative game* (a game, for short) is a pair (N, v), where $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$, is the characteristic function which assigns to each coalition S the worth $v(S)$ it can attain. If no confusion arises, a game (N, v) is denoted by simply v. For any coalition $S \subseteq N$, $N \setminus S = \{i \in N \mid i \notin S\}$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. A game (N, v) is zero-monotonic if for any pair of coalitions S, T, $S \subset T \subseteq N$, it holds $v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$. Let (N, v) and (N, w) be two games on the same player set, then $v \leq w$ if and only if $v(S) \leq w(S)$ for all $S \subseteq N$. A game is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all disjoint $S, T \subseteq N$. The *superadditive cover* of (N, v) is the minimum superadditive game (N, w) such that $v \leq w$. Two games (N, v) and (N, w) are strategically equivalent if and only if there exist $\alpha > 0$ and $d \in \mathbb{R}^N$ such that $w(S) = \alpha v(S) + \sum_{i \in S} d_i$. Given a vector $d \in \mathbb{R}^N$, the game (N, v) with characteristic function $v(S) = \sum_{i \in S} d_i$ is the *modular game* generated by d.

Given a game (N, v), a payoff vector is $x \in \mathbb{R}^N$, where x_i stands for the payoff to player $i \in N$. The restriction of a payoff vector to a coalition S is denoted by $x_{|S}$. By the standard notation, $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$, and $x(\emptyset) = 0$. Given $x, y \in \mathbb{R}^N$, we write $y \geq x$ if $y_i \geq x_i$ for all $i \in N$, and $y > x$ whenever $y \geq x$ and there exists $i \in N$ such that $y_i > x_i$. An *imputation* is a payoff vector x that is efficient, $x(N) = v(N)$, and individually rational, $x_i \geq v(\{i\})$ for all $i \in N$. The set of all imputations of a game (N, v) is denoted by $I(N, v)$, and when $I(N, v) \neq \emptyset$ the game is said to be *essential*. The *core*, denoted by $C(N, v)$, is the set of imputations that are efficient and coalitionally rational, that is, $x(S) \geq v(S)$ for all $S \subseteq N$. A game with a non-empty core is called *balanced*. Given a balanced game, a well-known single–valued core selection is *the nucleolus* (Schmeidler, 1969). The *excess* of a coalition S at an imputation $x \in I(N, v)$ is $e(S, x) = v(S) - x(S)$. Let $\theta(x) \in \mathbb{R}^{2^n - 2}$ be the *vector of excesses* of all coalitions (different from the grand coalition and the empty set) at x, arranged in a nonincreasing order. Then, the *nucleolus* of the game (N, v) is the imputation $\eta(v)$ that minimizes $\theta(x)$ with respect to the lexicographic order over the set of imputations: $\theta(\eta(v)) \leq_{Lex} \theta(x)$ for all $x \in I(N, v)$. This means that, for all $x \in I(N, v)$, either $\theta(\eta(v)) = \theta(x)$ or $\theta(\eta(v))_1 < \theta(x)_1$ or there exists $k \in \{1, 2, \ldots, 2^n - 3\}$ such that $\theta(\eta(v))_i = \theta(x)_i$ for all $1 \leq i \leq k$ and $\theta(\eta(v))_{k+1} < \theta(x)_{k+1}$.

It is well known that if (N, v) is balanced and its superadditive cover (N, w) satisfies $w(N) = v(N)$, then $C(N, w) = C(N, v)$. The analogous property is proved in Miquel and Núñez (2011) for the nucleolus: if a balanced game has the same efficiency level as its superadditive cover, then the nucleolus of both games coincide.

2.1. The assignment game

The assignment game is a coalitional form game that represents a two-sided market situation. An *assignment market* is a quintuple $\gamma = (M, M', A, p, q)$ where M and M' are two disjoint finite sets (the two sides of a market) of cardinality $|M| = m$ and $|M'| = m'$, $A = (a_{ij})_{(i,j)\in M\times M'}$ is a real $m \times m'$ matrix, and the vectors $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^{m'}$ are the reservation prices of buyers and sellers respectively. For all $(i, j) \in M \times M'$, the real number a_{ij} denotes the join profit obtained by the pair (i, j) if they trade.

If $S \subseteq M$ and $T \subseteq M'$, a *matching* μ between S and T is a bijection from a subset $Dom(\mu)$ of S, that we name the domain of μ, and a subset $Im(\mu)$ of T, that we name the image of μ. If $i \in S$ and $j \in T$ are related by μ we indistinctly write $(i, j) \in \mu$, $j = \mu(i)$ or $i = \mu^{-1}(j)$. We denote by $\mathcal{M}(S,T)$ the set of matchings between S and T. Given an assignment market $\gamma = (M, M', A, p, q)$, for all $S \subseteq M$, $T \subseteq M'$ and $\mu \in \mathcal{M}(M, M')$ we write

$$v(S,T;\mu) = \sum_{i\in Dom(\mu)} a_{i\mu(i)} + \sum_{i\in S\setminus Dom(\mu)} p_i + \sum_{j\in T\setminus Im(\mu)} q_j,$$

with the convention that any summation under an empty set of indices is zero.

A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the assignment market $\gamma = (M, M', A, p, q)$ if for all $\mu' \in \mathcal{M}(M, M')$ it holds $v(M, M'; \mu) \geq v(M, M'; \mu')$. The set of optimal matchings for the assignment market γ is denoted by $\mathcal{M}^*_\gamma(M, M')$. Similarly, $\mathcal{M}(S,T)$ and $\mathcal{M}^*_\gamma(S,T)$ denote respectively the set of matchings and optimal matchings of the submarket $\gamma' = (S, T, A_{|S\times T}, p_{|S}, q_{|T})$, where $A_{|S\times T}$ is the submatrix of A formed by rows corresponding to buyers in S and columns corresponding to sellers in T.

To any assignment market $\gamma = (M, M', A, p, q)$, a game in coalitional form (*assignment game*) is associated with player set $M \cup M'$ and characteristic function w_γ defined as follows: for all $S \subseteq M$ and $T \subseteq M'$,

$$w_\gamma(S \cup T) = max\,\{v(S,T;\mu) \mid \mu \in \mathcal{M}(S,T)\}\,.$$

This assignment game, that allows for agents' reservation prices, is a generalization of the classical assignment game of Shapley and Shubik (1972) (that is, an assignment game with non-negative matrix and null reservation prices) and has been considered for instance in Owen (1992) and Toda (2005). Given an assignment market γ, with some abuse of notation we sometimes also denote by γ its associated coalitional game. We denote by Γ_{AG} the set of assignment games. Following Owen (1992), we allow that one side of the market could be empty.[2] Unlike the set of Shapley and Shubik's classical assignment games, Γ_{AG} is closed by strategic equivalence. In fact, it can be shown that every assignment game is strategically equivalent to a classical assignment game in the sense of Shapley and Shubik.[3] As a consequence, Shapley and Shubik's results on the core of the assignment game extend to Γ_{AG}.

The core of the assignment game is always nonempty and it is formed by those efficient payoff vectors $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ that satisfy coalitional rationality for mixed-pair coalitions and

[2] If $\gamma = (M, p)$, then the associated assignment game (M, w_γ) is the modular game generated by the vector of reservation prices $p \in \mathbb{R}^M$, that is, $w_\gamma(S) = \sum_{i\in S} p_i$, for all $S \subseteq M$. Similarly, if $\gamma = (M', q)$, then $w_\gamma(S) = \sum_{i\in S} q_i$, for all $S \subseteq M'$.

[3] Let $\gamma = (M, M', A, p, q)$ be an assignment market where $A = (a_{ij})_{(i,j)\in M\times M'}$, $p \in \mathbb{R}^m$, $q \in \mathbb{R}^{m'}$, and let $\tilde{\gamma} = (M, M', \tilde{A})$ be an assignment market with null reservation prices and matrix $\tilde{A} = (\tilde{a}_{ij})_{(i,j)\in M\times M'}$ given by $\tilde{a}_{ij} := \max\{0, a_{ij} - p_i - q_j\}$, for all $(i, j) \in M \times M'$. Then, as the reader can easily check, $w_\gamma(S \cup T) = w_{\tilde{\gamma}}(S \cup T) + \sum_{i\in S} p_i + \sum_{j\in T} q_j$, for all $S \subseteq M$ and $T \subseteq M'$.

one-player coalitions:

$$C(M \cup M', w_\gamma) = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} \sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_\gamma(M \cup M'), \\ u_i + v_j \geq a_{ij} \text{ for all } (i,j) \in M \times M', \\ u_i \geq p_i \text{ for all } i \in M, \;\; v_j \geq q_j \text{ for all } j \in M'. \end{array} \right\}$$

If $\mu \in \mathcal{M}^*_\gamma(M, M')$, any core allocation $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ satisfies

$$u_i + v_j = a_{ij} \;\text{ for all } (i, j) \in \mu, \tag{1}$$

$$u_i = p_i \;\text{ for all } i \in M \setminus Dom(\mu), \tag{2}$$

$$v_j = q_j \;\text{ for all } j \in M' \setminus Im(\mu). \tag{3}$$

There exists a *buyer-optimal core allocation*, $(\bar{u}^\gamma, \underline{v}^\gamma)$, where each buyer attains her maximum core payoff and each seller his minimum one, and a *seller-optimal core allocation*, $(\underline{u}^\gamma, \bar{v}^\gamma)$, with the converse situation. By Roth and Sotomayor (1990),

$$\bar{u}^\gamma_i = w_\gamma(M \cup M') - w_\gamma((M \cup M') \setminus \{i\}) \quad \mathit{for\ all}\ i \in M, \tag{4}$$

and

$$\bar{v}^\gamma_j = w_\gamma(M \cup M') - w_\gamma((M \cup M') \setminus \{j\}) \quad \mathit{for\ all}\ j \in M'. \tag{5}$$

The *fair division point*, $\tau\left(w_\gamma\right)$, is defined by Thompson (1981) as the midpoint between these two extreme core allocations:

$$\tau\left(w_\gamma\right) = \frac{1}{2}(\bar{u}^\gamma, \underline{v}^\gamma) + \frac{1}{2}(\underline{u}^\gamma, \bar{v}^\gamma). \tag{6}$$

3. Derived consistency and another axiomatization of the core of the assignment game

We begin introducing the concept of a solution on the domain of assignment games. The next definitions follow Toda (2005).

Definition 3.1. *Let $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$. A payoff vector $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is* feasible *if there exists $\mu \in \mathcal{M}(M, M')$ such that*

(i) *$u_i = p_i$ for all $i \in M \setminus Dom(\mu)$, $v_j = q_j$ for all $j \in M' \setminus Im(\mu)$, and*

(ii) *$u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$.*

In the above definition, μ is said to be *compatible* with (u, v).

Definition 3.2. *A* solution *on Γ_{AG} is a correspondence σ that associates a nonempty subset of feasible payoff vectors with each $\gamma \in \Gamma_{\mathrm{AG}}$.*

Consistency is a standard property used to analyze the behavior of solutions with respect to reduction of population.[4] To introduce consistency, first we need to define the concept of a reduced game. The terminology is taken from Thomson (2006).

[4]For a comprehensive survey on the consistency principles, the reader is referred to Thomson (2006).

Definition 3.3. *Let (N, v) be a game, $x \in \mathbb{R}^N$ and $\emptyset \neq T \subset N$.*

1. *The* **max reduced game** *(Davis and Maschler, 1965) relative to T at x is the game $\left(T, r^{\mathrm{DM}}_{T,x}(v)\right)$ defined by*

$$r^{\mathrm{DM}}_{T,x}(v)(S) = \begin{cases} v(N) - x(N \setminus T) & \text{if} \quad S = T, \\ \max_{Q \subseteq N \setminus T}\{v(S \cup Q) - x(Q)\} & \text{if} \quad \emptyset \neq S \subset T, \\ 0 & \text{if} \quad S = \emptyset. \end{cases} \tag{7}$$

2. *The* **projected reduced game** *relative to T at x is the game $(T, r_{T,x}(v))$ defined by*

$$r_{T,x}(v)(S) = \begin{cases} v(N) - x(N \setminus T) & \text{if} \quad S = T, \\ v(S) & \text{if} \quad S \subset T. \end{cases} \tag{8}$$

Let σ be a solution on the domain of coalitional games Γ. We say that σ satisfies

- **max consistency** if for all $(N, v) \in \Gamma$, all $\emptyset \neq T \subset N$ and all $x \in \sigma(N, v)$, then $\left(T, r^{\mathrm{DM}}_{T,x}(v)\right) \in \Gamma$ and $x_{|T} \in \sigma\left(T, r^{\mathrm{DM}}_{T,x}(v)\right)$.
- **projection consistency** if for all $(N, v) \in \Gamma$, all $\emptyset \neq T \subset N$ and all $x \in \sigma(N, v))$, then $(T, r_{T,x}(v) \in \Gamma$ and $x_{|T} \in \sigma(T, r_{T,x}(v)$.

Peleg (1986) uses max consistency to characterize the core on the domain of all coalitional form games. The nucleolus is known to be max consistent in the class of zero-monotonic games (Potters, 1991). The core is also projection consistent on the domain of coalitional games.

On the domain of assignment games, it is immediate to notice that given a feasible payoff vector (u, v), the projection reduced game may not be an assignment game. Because of this, Toda (2005) slightly modifies the definition of projection reduced game for the class of assignment games.

Definition 3.4. *Let $\gamma = (M, M', A, p, q)$ be an assignment market, $x = (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ feasible and μ a matchimg compatible with x. Let $\emptyset \neq T \subseteq M \cup M'$ such that $\mu(M \cap T) = M' \cap T$. The* projection reduced assignment market (Toda, 2005) *relative to T at x is*

$$\gamma' = (T \cap M, T \cap M', A_{|(T \cap M) \times (T \cap M')}, p_{|T \cap M}, q_{|T \cap M'}) .$$

The **projection reduced assignment game** *relative to T at x is the coalitional game associated to the projection reduced assignment market γ' and it will be denoted by $\left(T, pr_{T,x}(w_\gamma)\right)$.*

A solution σ on Γ_{AG} satisfies

- **projection consistency** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$, all $\emptyset \neq T \subset M \cup M'$ and all $x \in \sigma\left(M \cup M', w_\gamma\right)$, then $\left(T, pr_{T,x}(w_\gamma)\right) \in \Gamma_{\mathrm{AG}}$ and $x_{|T} \in \sigma\left(T, pr_{T,x}(w_\gamma)\right)$.

It is straightforward to see that the core is projection consistent on the domain of assignment games.

As for max consistency, it turns out again that the max reduced game of an assignment game may not belong to the class of assignment games. To overcome this drawback, we introduce Owen's derived game.

Definition 3.5. *Let $\gamma = (M, M', A, p, q)$ be an assignment market, $\emptyset \neq T \subset M \cup M'$, and $x = (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$. The* derived assignment market (Owen, 1992) *relative to T at x is $\hat{\gamma} = (T \cap M, T \cap M', A_{|(T\cap M)\times(T\cap M')}, \hat{p}, \hat{q})$, where*

$$\hat{p}_i = \max\left\{p_i, \max_{j\in M'\setminus T}\{a_{ij} - v_j\}\right\}, \textit{ for all } i \in T \cap M,$$

$$\hat{q}_j = \max\left\{q_j, \max_{i\in M\setminus T}\{a_{ij} - u_i\}\right\}, \textit{ for all } j \in T \cap M'.$$

The **derived assignment game** *relative to T at x is the coalitional game associated to the derived assignment market $\hat{\gamma}$ and it will be denoted by $\left(T, d_{T,x}(w_\gamma)\right)$.*

The main result in Owen (1992) states that, if $x \in C(M \cup M', w_\gamma)$, then the derived game $\left(T, d_{T,x}(w_\gamma)\right)$ is the superadditive cover of the max reduced game $\left(T, r^{\mathrm{DM}}_{T,x}(w_\gamma)\right)$.[5]

Next we define consistency with respect to this derived game. A solution σ on Γ_{AG} satisfies

- **derived consistency** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$, all $\emptyset \neq T \subset M \cup M'$ and all $x \in \sigma\left(M \cup M', w_\gamma\right)$, then $\left(T, d_{T,x}(w_\gamma)\right) \in \Gamma_{\mathrm{AG}}$ and $x_{|T} \in \sigma\left(T, d_{T,x}(w_\gamma)\right)$.

The reader will easily check that the core satisfies derived consistency on the domain of assignment games.

Besides the above consistency axioms, a desirable property for a solution on a class of balanced games, as it is the case of assignment games, is that it lies in the core. A solution σ on Γ_{AG} satisfies

- **core selection** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\mathrm{AG}}$, $\sigma(M \cup M', w_\gamma) \subseteq C(M \cup M', w_\gamma)$.

We first prove that, on the domain Γ_{AG}, any solution satisfying derived consistency selects core elements.

Proposition 3.1. *On the domain of assignment games, derived consistency implies core selection.*

Proof. Let σ be a solution on Γ_{AG} satisfying derived consistency. Let it be $\gamma = (M, M', A, p, q)$ and $z = (u, v) \in \sigma(M \cup M', w_\gamma)$. If $M \neq \emptyset$ and $M' = \emptyset$, then the game (M, w_γ) is the modular game generated by the vector $p \in \mathbb{R}^M$. Feasibility of the solution (Definition 3.1) implies $z = p$ and thus $C(M, w_\gamma) = \{z\}$. Similarly, if $M = \emptyset$ and $M' \neq \emptyset$, then $z = q$ and $C(M', w_\gamma) = \{z\}$.

Assume now that $M \neq \emptyset$ and $M' \neq \emptyset$. For all $i \in M$ consider the derived game relative to $T = \{i\}$ at z. By derived consistency, $u_i \in \sigma(\{i\}, d_{\{i\},z}(w_\gamma))$ and by Definitions 3.1 and 3.2, $u_i = \hat{p}_i = \max\{p_i, \max_{j\in M'}\{a_{ij} - v_j\}\}$, which implies that, for all $i \in M$, $u_i \geq p_i$ and $u_i + v_j \geq a_{ij}$ for all $j \in M'$. Similarly, for all $j \in M'$ let us consider the derived game relative to $T = \{j\}$ at z. Again by derived consistency, $v_j \in \sigma(\{j\}, d_{T,z}(w_\gamma))$ and by Definitions 3.1 and 3.2, $v_j = \hat{q}_j = \max\{q_j, \max_{i\in M}\{a_{ij} - u_i\}\}$ which implies $v_j \geq q_j$ for all $j \in M'$. Hence, $z = (u, v)$ satisfies coalitional rationality for all mixed-pair and individual coalitions. It only remains to check its efficiency.

[5]See Owen (1992) and Thomson (2006) for an interpretation of the derived game.

Let $\mu \in \mathcal{M}^*_\gamma(M, M')$ be an optimal matching and $\mu' \in \mathcal{M}(M, M')$ another matching that is compatible with $z = (u, v)$. Notice that such μ' exists since $z = (u, v)$ is feasible by Definition 3.2. Then,

$$\begin{aligned}\sum_{i \in Dom(\mu)} a_{i\mu(i)} + \sum_{i \in M \setminus Dom(\mu)} p_i + \sum_{j \in M' \setminus Im(\mu)} q_j &\leq \sum_{i \in Dom(\mu)} \left(u_i + v_{\mu(i)}\right) + \sum_{i \in M \setminus Dom(\mu)} u_i + \sum_{j \in M' \setminus Im(\mu)} v_j \\ &= \sum_{i \in Dom(\mu')} \left(u_i + v_{\mu'(i)}\right) + \sum_{i \in M \setminus Dom(\mu')} u_i + \sum_{j \in M' \setminus Im(\mu')} v_j \\ &= \sum_{i \in Dom(\mu')} a_{i\mu'(i)} + \sum_{i \in M \setminus Dom(\mu')} p_i + \sum_{j \in M' \setminus Im(\mu')} q_j.\end{aligned}$$

By optimality of μ, the above inequality implies

$$\sum_{i \in Dom(\mu)} a_{i\mu(i)} + \sum_{i \in M \setminus Dom(\mu)} p_i + \sum_{j \in M' \setminus Im(\mu)} q_j = \sum_{i \in Dom(\mu')} a_{i\mu'(i)} + \sum_{i \in M \setminus Dom(\mu')} p_i + \sum_{j \in M' \setminus Im(\mu')} q_j$$

and thus $\sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_\gamma(M \cup M')$ which concludes the proof of $z = (u, v) \in C(M \cup M', w_\gamma))$. □

Making use of the above proposition, we characterize the core on the domain of assignment games only in terms of two consistency axioms.

Theorem 3.1. *On the domain of assignment games, the only solution satisfying derived consistency and projection consistency is the core.*

Proof. It only remains to prove uniqueness. Let σ be a solution on Γ_{AG} satisfying derived consistency and projection consistency. By Proposition 3.1, σ is a subcorrespondence of the core. And from Lemma 3.1 in Toda (2005), every subcorrespondence of the core that satisfies projection consistency coincides with the core. □

4. An axiomatic characterization of the nucleolus of the assignment game

In this section, we characterize axiomatically the nucleolus by means of two axioms: *derived consistency* with respect to the reduced game introduced by Owen (1992), and *equal maximum complaint between groups*. As a by-product, we characterize the locus of the nucleolus within the core.

Due to the bilateral feature of the market, we look for an axiom that guarantees some stability between groups. A not much demanding property is the equal maximum complaint between both sides of the market that requires that the individual excess of the most dissatisfied buyer (seller) is no less that the individual excess of the most dissatisfied seller (buyer).

A solution σ on Γ_{AG} satisfies

- **equal maximum complaint between groups** if for all $\gamma = (M, M', A, p, q) \in \Gamma_{\text{AG}}$ with $|M| = |M'|$ and all $x \in \sigma(M \cup M', w_\gamma)$, then

$$\max_{i \in M}\{e(\{i\}, x)\} = \max_{j \in M'}\{e(\{j\}, x)\}. \tag{9}$$

Next we prove that the nucleolus satisfies derived consistency and equal maximum complaint between groups.

Proposition 4.1. *On the domain of assignment games, the nucleolus satisfies derived consistency.*

Proof. Let it be $\gamma = (M, M', A, p, q)$, $\emptyset \neq T \subset M \cup M'$ and let $\eta = \eta(w_\gamma)$ be the nucleolus of the assignment game $(M \cup M', w_\gamma)$. Let $\left(T, r^{\mathrm{DM}}_{T,\eta}(w_\gamma)\right)$ and $\left(T, d_{T,\eta}(w_\gamma)\right)$ be the max reduced game and the derived game relative to T at η, respectively. By Potters (1991) we know that the nucleolus of the assignment game satisfies $\eta_{|T} = \eta\left(r^{\mathrm{DM}}_{T,\eta}(w_\gamma)\right)$.[6] Since $\eta \in C(M \cup M', w_\gamma)$, by Owen (1992) we know that the derived game $(T, d_{T,\eta}(w_\gamma))$ is the superadditive cover of the max reduced game. Moreover, it is known from Miquel and Núñez (2011) that if a balanced coalitional game and its superadditive cover have the same efficiency level, then they have the same nucleolus. Thus, we only need to prove that $r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(T) = d_{T,\eta}(w_\gamma)(T)$ to obtain $\eta_{|T} = \eta\left(d_{T,\eta}(w_\gamma)\right)$.

Since $(T, d_{T,\eta}(w_\gamma))$ is the superadditive cover of $\left(T, r^{\mathrm{DM}}_{T,\eta}(w_\gamma)\right)$, $r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(T) \leq d_{T,\eta}(w_\gamma)(T)$. To see the converse inequality notice the following: for all $i \in T \cap M$, it follows from (7) that $r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(\{i\}) \geq w_\gamma(\{i\}) = p_i$ and also $r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(\{i\}) = \max_{Q \subseteq (M \cup M') \setminus T} \left\{w_\gamma(\{i\}) \cup Q) - \eta(Q)\right\} \geq \max_{j \in M' \setminus T} \left\{a_{ij} - \eta_j\right\}$. Hence,

$$r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(\{i\}) \geq \max\left\{p_i, \max_{j \in M' \setminus T}\{a_{ij} - \eta_j\}\right\} = \hat{p}_i. \tag{10}$$

Similarly, for all $j \in T \cap M'$,

$$r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(\{j\}) \geq \hat{q}_j. \tag{11}$$

Now, let $\mu \in \mathcal{M}(T \cap M, T \cap M')$ be an optimal matching for the derived assignment game $(T, d_{T,\eta}(w_\gamma))$. Then,

$$\begin{aligned}
d_{T,\eta}(w_\gamma)(T) &= \sum_{i \in Dom(\mu)} a_{i\mu(i)} + \sum_{i \in (T \cap M) \setminus Dom(\mu)} \hat{p}_i + \sum_{j \in (T \cap M') \setminus Im(\mu)} \hat{q}_j \\
&\leq \sum_{i \in Dom(\mu)} a_{i\mu(i)} + \sum_{i \in (T \cap M) \setminus Dom(\mu)} r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(\{i\}) + \sum_{j \in (T \cap M') \setminus Im(\mu)} r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(\{j\}) \\
&\leq \sum_{i \in Dom(\mu)} \left(\eta_i + \eta_{\mu(i)}\right) + \sum_{i \in (T \cap M) \setminus Dom(\mu)} \eta_i + \sum_{j \in (T \cap M') \setminus Im(\mu)} \eta_j \\
&= \sum_{k \in T} \eta_k = r^{\mathrm{DM}}_{T,\eta}(w_\gamma)(T),
\end{aligned}$$

where the first inequality follows from (10) and (11), the second inequality follows from the fact that $\eta \in C(M \cup M', w_\gamma)$ and $\eta_{|T} \in C\left(T, r^{\mathrm{DM}}_{T,\eta}(w_\gamma)\right)$, and the last equality from efficiency of the nucleolus $\eta_{|T}$ of the max reduced game. □

Proposition 4.2. *On the domain of assignment games, the nucleolus satisfies equal maximum complaint between groups.*

Proof. Let it be $\gamma = (M, M', A, p, q)$ with $|M| = |M'|$ and let $\eta = \eta(w_\gamma)$ be the nucleolus of the assignment game $(M \cup M', w_\gamma)$. Let $\varepsilon_1 = \max_{i \in M}\{e(\{i\}, \eta)\}$, $\varepsilon_2 = \max_{j \in M'}\{e(\{j\}, \eta)\}$ and assume, without loss of generality, that $\varepsilon_1 < \varepsilon_2$. Now define the payoff vectors

$$(u', v') = (\eta_{|M} - \varepsilon_1 \cdot e^M, \eta_{|M'} + \varepsilon_1 \cdot e^{M'}) \text{ and } (u'', v'') = (\eta_{|M} + \varepsilon_2 \cdot e^M, \eta_{|M'} - \varepsilon_2 \cdot e^{M'}), \tag{12}$$

[6] However, Owen (1992) provides an example showing that, in general, the max reduced $\left(T, r^{\mathrm{DM}}_{T,\eta}(w_\gamma)\right)$ need not be an assignment game.

where $e^M = (1, \ldots, 1) \in \mathbb{R}^M$ and $e^{M'} = (1, \ldots, 1) \in \mathbb{R}^{M'}$.

It can be easily checked that $(u', v'), (u'', v'') \in C(M \cup M', w_\gamma)$. Now take $z = (u, v) = \frac{1}{2}(u', v') + \frac{1}{2}(u'', v'')$. By substitution from (12), for all $(i, j) \in M \times M'$,

$$\begin{aligned} e(\{i, j\}, z) &= a_{ij} - u_i - v_j = a_{ij} - \left(\eta_i + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\right) - \left(\eta_j + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)\right) \\ &= a_{ij} - \eta_i - \eta_j = e(\{i, j\}, \eta). \end{aligned}$$

Thus, to lexicographically minimize the vector of ordered excesses we only need to consider excesses of individual coalitions. First,

$$\begin{aligned} \max_{k \in M \cup M'} \{e(\{k\}, z)\} &= \max \left\{ \max_{k \in M} \{p_k - u_k\}, \max_{k \in M'} \{q_k - v_k\} \right\} \\ &= \max \left\{ \max_{k \in M} \{p_k - \eta_k\} + \frac{1}{2}(\varepsilon_2 - \varepsilon_1), \max_{k \in M'} \{q_k - \eta_k\} + \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \right\} \\ &= \max \left\{ \varepsilon_1 + \frac{1}{2}(\varepsilon_2 - \varepsilon_1), \varepsilon_2 + \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \right\} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2), \end{aligned}$$

where the second equality follows from (12). Moreover, $\varepsilon_2 = \max_{k \in M \cup M'} \{e(\{k\}, \eta)\}$ since $\varepsilon_1 < \varepsilon_2$. But then, $\max_{k \in M \cup M'} \{e(\{k\}, z)\} = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) < \varepsilon_2 = \max_{k \in M \cup M'} \{e(\{k\}, \eta)\}$, in contradiction with η being the nucleolus. Hence, $\varepsilon_1 = \varepsilon_2$ and this concludes the proof. □

We are now in disposition to state and prove the axiomatic characterization of the nucleolus.

Theorem 4.1. *On the domain of assignment games, the only solution satisfying derived consistency and equal maximum complaint between groups is the nucleolus.*

Proof. From Propositions 4.1 and 4.2 we know that the nucleolus satisfies both properties. To show uniqueness assume there exists a solution σ on Γ_{AG} satisfying derived consistency and equal maximum complaint between groups.

Let it be $\gamma = (M, M', A, p, q)$ and $z = (u, v) \in \sigma(M \cup M', w_\gamma)$. From Proposition 3.1, σ satisfies core selection and thus $z \in C(M \cup M', w_\gamma)$. If $M \neq \emptyset$ and $M' = \emptyset$ (or $M = \emptyset$ and $M' \neq \emptyset$) the assignment game is a modular game generated by p (or q) and from Definition 3.2 we have $z = \eta$. Assume then $M \neq \emptyset$, $M' \neq \emptyset$ and $z \neq \eta$.

Let $\mu \in \mathcal{M}^*_\gamma(M, M')$ be an optimal matching. For any $\emptyset \neq S \subseteq M$ such that $|S| = |\mu(S)|$ let us consider the derived game relative to $T = S \cup \mu(S)$ at z. By derived consistency of σ, $z_{|T} \in \sigma(T, d_{T,z}(w_\gamma))$. Since $|S| = |\mu(S)|$, by equal maximum complaint between groups of σ applied to the derived game $(T, d_{T,z}(w_\gamma))$, we have

$$\max_{i \in S} \{e(\{i\}, z)\} = \max_{j \in \mu(S)} \{e(\{j\}, z)\}, \tag{13}$$

where $e(\{i\}, z) = d_{T,z}(w_\gamma)(\{i\}) - z_i = \hat{p}_i - z_i$ and $e(\{j\}, z) = d_{T,z}(w_\gamma)(\{j\}) - z_j = \hat{q}_j - z_j$, for all $i \in S$ and all $j \in \mu(S)$. From the definition of $\hat{p}_i$, we obtain

$$\max_{i \in S} \{e(\{i\}, z)\} = \max_{i \in S} \left\{ \max \left\{ p_i, \max_{k \in M' \setminus \mu(S)} \{a_{ik} - z_k\} \right\} - z_i \right\} = \max_{\substack{i \in S \\ k \in M' \setminus \mu(S)}} \{p_i - z_i, a_{ik} - z_i - z_k\}. \tag{14}$$

Similarly, making use of the definition of $\hat{q}_j$,

$$\max_{j \in \mu(S)} \{e(\{i\}, z)\} = \max_{\substack{j \in \mu(S) \\ k \in M \setminus S}} \{q_j - z_j, a_{kj} - z_j - z_k\}, \tag{15}$$

and, as a consequence of (14) and (15), expression (13) is equivalent to

$$\max_{\substack{i\in S\\ k\in M'\setminus\mu(S)}} \{p_i - z_i, a_{ik} - z_i - z_k\} = \max_{\substack{j\in\mu(S)\\ k\in M\setminus S}}\{q_j - z_j, a_{kj} - z_j - z_k\}, \tag{16}$$

for all $S \subseteq M$ with $|S| = |\mu(S)|$ and all $z \in \sigma(M \cup M', w_\gamma)$, being σ a solution satisfying derived consistency and equal maximum complaint between groups. Since the nucleolus also satisfies these two axioms, we have

$$\max_{\substack{i\in S\\ k\in M'\setminus\mu(S)}} \{p_i - \eta_i, a_{ik} - \eta_i - \eta_k\} = \max_{\substack{j\in\mu(S)\\ k\in M\setminus S}}\{q_j - \eta_j, a_{kj} - \eta_j - \eta_k\}, \tag{17}$$

for all $S \subseteq M$ with $|S| = |\mu(S)|$. Now, from $z \neq \eta$, either there exists a nonempty coalition $S \subseteq M$ such that $z_i > \eta_i$ for all $i \in S$ and $z_i \leq \eta_i$ for all $i \in M \setminus S$, or there exists a nonempty coalition $S \subseteq M$ such that $z_i < \eta_i$ for all $i \in S$ and $z_i \geq \eta_i$ for all $i \in M \setminus S$. Let us assume without loss of generality that the first case holds, since the proof in the second case is analogous. From $z \in C(M \cup M', w_\gamma)$ follows $z_j < \eta_j$ for all $j \in \mu(S)$ and $z_j \geq \eta_j$ for all $j \in M' \setminus \mu(S)$. Moreover, all agents in S are matched by μ. Indeed, if $i \in S \setminus Dom(\mu)$, from $z \in C(M \cup M', w_\gamma)$ we have $z_i = p_i > \eta_i$, in contradiction with the nucleolus being in the core. Then,

$$\begin{aligned}\max_{\substack{i\in S\\ j\in M'\setminus\mu(S)}} \{p_i - z_i, a_{ij} - z_i - z_j\} &< \max_{\substack{i\in S\\ j\in M'\setminus\mu(S)}} \{p_i - \eta_i, a_{ij} - \eta_i - \eta_j\}\\ &= \max_{\substack{j\in\mu(S)\\ i\in M\setminus S}} \{q_j - \eta_j, a_{ij} - \eta_i - \eta_j\}\\ &< \max_{\substack{j\in\mu(S)\\ i\in M\setminus S}} \{q_j - z_j, a_{ij} - z_i - z_j\},\end{aligned} \tag{18}$$

in contradiction with (16). Hence, $z = \eta$. □

The following examples show that in Theorem 4.1 the axioms are independent:

Example 4.1. *(A solution violating equal maximum complaint between groups).*
For each $\gamma = (M, M', A, p, q)$, let $\sigma_1(M \cup M', w_\gamma) = C(M \cup M', w_\gamma)$. We already know that the core satisfies derived consistency on Γ_{AG}. To see that the core does not satisfy equal maximum complaint between groups, take for instance the assignment market $\gamma = (\{1, 2\}, \{1', 2'\}, A, 0, 0)$ where the assignment matrix is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Take the feasible payoff vector $z = (\overline{u}^\gamma, \underline{v}^\gamma) = (1, 1; 0, 0) \in C(M \cup M', w_\gamma)$. Notice that $\max_{i\in M}\{e(\{i\}, z)\} = -1 \neq 0 = \max_{j\in M'}\{e(\{j\}, z)\}$.

Before presenting the next example, let us remark that, equal maximum complaint between groups could also be interpreted in geometric terms when combined with core selection. Notice first that saying that solution σ on Γ_{AG} satisfies (9) is equivalent to saying

$$\min_{i\in M}\{u_i - p_i\} = \min_{j\in M'}\{v_j - q_j\}, \tag{19}$$

for all $x = (u, v) \in \sigma(M \cup M', w_\gamma)$. Suppose now that $\sigma(M \cup M', w_\gamma) \subseteq C(M \cup M', w_\gamma)$. For all $S \subseteq M$, let the vector $e^S \in \mathbb{R}^M$ be defined by $(e^S)_i = 1$ for all $i \in S$ and $(e^S)_i = 0$ for all $i \in M \setminus S$. The vector $e^T \in \mathbb{R}^{M'}$, for all $T \subseteq M'$, is defined analogously. Take $\varepsilon_1(u, v) = \min_{i\in M}\{u_i - p_i\}$ and notice that $\varepsilon_1(u, v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon \cdot e^M, v + \varepsilon \cdot e^{M'}) \in C(M \cup M', w_\gamma)\}$. The reason is that, for all $\varepsilon \geq 0$, efficiency and coalitional rationality for mixed-pair coalitions holds trivially for the payoff vector $(u - \varepsilon \cdot e^M, v + \varepsilon \cdot e^{M'})$ and, as long as $\varepsilon \leq \varepsilon_1(u, v)$, individual rationality also holds. Similarly, if we write $\varepsilon_2(u, v) = \min_{j\in M'}\{v_j - q_j\}$, we can check that $\varepsilon_2(u, v) = \max\{\varepsilon \geq$

$0 \mid (u+\varepsilon\cdot e^M, v-\varepsilon\cdot e^{M'}) \in C(M\cup M', w_\gamma)\}$. As a consequence, if σ satisfies core selection, then for each chosen allocation $(u,v)\in\mathbb{R}^M\times\mathbb{R}^{M'}$, the largest per capita amount that sector M can transfer to sector M' without leaving the core equals the largest per capita amount that sector M' can transfer to sector M without getting outside the core. This means that core elements satisfying (9) are at the midpoint of a certain 45-slope range within the core. Next Example 4.2 shows that this bisection property between sectors also holds for the tau value.

Example 4.2. *(A solution violating derived consistency).*
For each $\gamma = (M, M', A, p, q)$, let $\sigma_2(M\cup M', w_\gamma) = \tau(w_\gamma)$, as defined in (6). Denote $\tau = \tau(w_\gamma)$. To check equal maximum complaint between groups suppose $|M| = |M'|$. Recall that conditions (9) and (19) are equivalent. If there exists $\mu \in \mathcal{M}^*_\gamma(M,M')$ such that $Dom(\mu)\neq M$, then $\min_{i\in M}\{\tau_i - p_i\} = \min_{j\in M'}\{\tau_j - q_j\} = 0$. Assume then $Dom(\mu) = M$ for all $\mu\in\mathcal{M}^*_\gamma(M,M')$. As we have noted before, $\varepsilon_1 = \min_{i\in M}\{\tau_i - p_i\}$ is the largest amount to guarantee that the vector $z = \left(\tau_{|M} - \varepsilon_1\cdot e^M, \tau_{|M'} + \varepsilon_1\cdot e^{M'}\right)$ remains in the core. Equivalently, we could impose $z_i = \tau_i - \varepsilon_1 \geq \underline{u}_i$ for all $i\in M$, and get, for any $\mu\in\mathcal{M}^*_\gamma(M,M')$, $\varepsilon_1 \leq \tau_i - \underline{u}_i = a_{i\mu(i)} - \tau_{\mu(i)} - \left(a_{i\mu(i)} - \bar{v}_{\mu(i)}\right) = \bar{v}_{\mu(i)} - \tau_{\mu(i)}$, where the first equality comes from the fact that τ and $(\underline{u},\bar{v})$ are core allocations. Thus, $\min_{i\in M}\{\tau_i - p_i\} = \min_{j\in M'}\{\bar{v}_j - \tau_j\}$. A symmetric argument, taking into account that $\varepsilon_2 = \min_{j\in M'}\{\tau_j - q_j\}$ is the largest amount to guarantee $\left(\tau_{|M} + \varepsilon_2\cdot e^M, \tau_{|M'} - \varepsilon_2\cdot e^{M'}\right) \in C\left(M\cup M', w_\gamma\right)$, leads to $\min_{j\in M'}\{\tau_j - q_j\} = \min_{i\in M}\{\bar{u}_i - \tau_i\}$. Thus, equal maximum complaint between groups, when applied to the tau value, can be written as $\min_{i\in M}\{\bar{u}_i - \tau_i\} = \min_{j\in M'}\{\bar{v}_j - \tau_j\}$ or, equivalently, by (4) and (5) we obtain

$$\max_{i\in M}\left\{w_\gamma(N\setminus\{i\}) + \tau_i\right\} = \max_{j\in M'}\left\{w_\gamma(N\setminus\{j\}) + \tau_j\right\}, \tag{20}$$

where $N = M\cup M'$.

We now check that τ satisfies (20). From expressions (4), (5) and (6), for all $i\in M$ and all $\mu\in\mathcal{M}^*_\gamma(M,M')$,

$$\begin{aligned}\tau_i &= \frac{w_\gamma(N) - w_\gamma(N\setminus\{i\}) + w_\gamma(N\setminus\{\mu(i)\}) - w_\gamma(N\setminus\{i,\mu(i)\})}{2}\\ \tau_{\mu(i)} &= \frac{w_\gamma(N) - w_\gamma(N\setminus\{\mu(i)\}) + w_\gamma(N\setminus\{i\}) - w_\gamma(N\setminus\{i,\mu(i)\})}{2}.\end{aligned}$$

Thus, for all $i\in M$

$$\begin{aligned}w_\gamma(N\setminus\{i\}) + \tau_i &= w_\gamma(N\setminus\{i\}) + \frac{w_\gamma(N) - w_\gamma(N\setminus\{i\}) + w_\gamma(N\setminus\{\mu(i)\}) - w_\gamma(N\setminus\{i,\mu(i)\})}{2}\\ &= \frac{w_\gamma(N) + w_\gamma(N\setminus\{i\}) + w_\gamma(N\setminus\{\mu(i)\}) - w_\gamma(N\setminus\{i,\mu(i)\})}{2}\\ &= w_\gamma(N\setminus\{\mu(i)\}) + \tau_{\mu(i)},\end{aligned}$$

which concludes that τ satisfies equal maximum complaint between groups.

5. A geometric characterization of the nucleolus of the assignment game

The *kernel* (Davis and Maschler, 1965) is another set-solution concept for cooperative games. The kernel, $\mathcal{K}(v)$, of an essential cooperative game (N,v) is always nonempty and it contains the nucleolus. For *zero–monotonic games*,[7] as it is the case of assignment games, the kernel can be described by

$$\mathcal{K}(v) = \{z \in I(v) \mid s^v_{ij}(z) = s^v_{ji}(z) \text{ for all } i,\ j \in N\,,\ i \neq j\}\,,$$

where the maximum surplus $s^v_{ij}(z)$ of player i over another player j with respect to the imputation z is defined by

$$s^v_{ij}(z) = \max\{\, e^v(S, z) | \, S \subseteq N\,,\ i \in S\,,\ j \notin S\,\}\ .$$

We will just write $s_{ij}(z)$ when no confusion regarding the game v can arise.

Given an arbitrary coalitional game (N, v), with any core allocation $z \in C(v)$ and any pair of agents $i, j \in N$, there is associated a non-negative real number $\delta^v_{ij}(z)$ designating the largest amount that can be transferred from player i to player j with respect to the core allocation z while remaining in the core of the game (N, v):

$$\delta^v_{ij}(z) = \max\{\varepsilon \geq 0 \mid z - \varepsilon e^i + \varepsilon e^j \in C(v)\},$$

where, for all $i \in N$, $e^i \in \mathbb{R}^N$ is the vector defined by $e^i_i = 1$ and $e^i_k = 0$ for all $k \neq i$, $k \in N$. This critical number $\delta^v_{ij}(z)$ was introduced by Maschler et al. (1979). For any core element $z \in C(v)$, this number $\delta^v_{ij}(z)$ is related to the excess $s^v_{ij}(z)$ in the definition of the kernel by $\delta^v_{ij}(z) = -s^v_{ij}(z)$. They prove in the aforementioned paper that a bisection property characterizes those elements in the intersection of the kernel and the core: $z \in C(v) \cap \mathcal{K}(v)$ if and only if z is the midpoint of the core segment with extreme points $z - \delta^v_{ij}(z)e^i + \delta^v_{ij}(z)e^j$ and $z + \delta^v_{ji}(z)e^i - \delta^v_{ji}(z)e^j$, for all $i, j \in N$. In this section we introduce a stronger bisection property that characterizes the nucleolus of the assignment game.

As for the kernel of assignment games, it turns out that it is always included in the core, $\mathcal{K}(w_A) \subseteq C(w_A)$ (Driessen, 1998). Moreover, if $(u, v) \in C(w_A)$, then *(a)* $s_{ij}(z) = 0$ whenever $i, j \in M$ or $i, j \in M'$, and *(b)* if $i \in M$ and $j \in M'$, then $s_{ij}(z)$ is always attained at the excess of some individual coalition or mixed–pair coalition:

$$s_{ij}(u, v) = \max_{k \in M' \setminus \{j\}} \{-u_i, a_{ik} - u_i - v_k\}.$$

As a consequence, given $(u, v) \in C(w_A)$, we get that $(u, v) \in \mathcal{K}(w_A)$ if and only if $s_{ij}(u, v) = s_{ji}(u, v)$ for all (i, j) belonging to all the optimal matchings, since the remaining equalities hold trivially.

The above proof of Theorem 4.1's suggest that the locus of the nucleolus inside the core can be determined by requiring this bisection property be satisfied not only for optimally matched pairs but also for optimally matched coalitions.

By adding dummy players, that is, null rows or columns in the assignment matrix and null reservation prices, we can assume from now on, without loss of generality, that the number of buyers equals the number of sellers, since this does not modify the nucleolus payoff of the non-dummy agents.[8]

Let $\gamma = (M, M', A, p, q)$ be an assignment market with $|M| = |M'|$. For each $\emptyset \neq S \subseteq M$, $\emptyset \neq T \subseteq M'$, $|S| = |T|$, we define *the largest amount that can be transferred from players in S to*

[7] A game (N, v) is zero-monotonic if for any pair of coalitions S, T, $S \subset T \subseteq N$ it holds $v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$.

[8] See Núñez (2004) for a detailed argument.

players in T with respect to the core allocation (u, v) *while remaining in the core of* $(M \cup M', w_\gamma)$ by

$$\delta^{w_\gamma}_{S,T}(u, v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon e^S, v + \varepsilon e^T) \in C(M \cup M', w_\gamma)\}. \tag{21}$$

Similarly,

$$\delta^{w_\gamma}_{T,S}(u, v) = \max\{\varepsilon \geq 0 \mid (u + \varepsilon e^S, v - \varepsilon e^T) \in C(M \cup M', w_\gamma)\}. \tag{22}$$

We write $\delta_{S,T}(u, v)$ and $\delta_{T,S}(u, v)$, respectively, if no confusion arises regarding the assignment game $(M \cup M', w_\gamma)$.

Notice that if there exists an optimal matching $\mu \in \mathcal{M}^*_\gamma(M, M')$ such that S and T do not correspond each other by this optimal matching ($\mu(S) \neq T$), then $\delta_{S,T}(u, v) = \delta_{T,S}(u, v) = 0$ for all $(u, v) \in C(M \cup M', w_\gamma)$. The reason is that if there exists $i \in S$ such that $\mu(i) \notin T$ (and similarly for $j \in T$ such that $\mu^{-1}(j) \notin S$) we have that the payoff vector $(u', v') = (u - \varepsilon e^S, v + \varepsilon e^T)$ will lie outside the core for all $\varepsilon > 0$. Indeed, if $i \notin Dom(\mu)$, then $u'_i = u_i - \varepsilon = p_i - \varepsilon < p_i$. Otherwise, $u'_i + v'_{\mu(i)} = u_i - \varepsilon + v_{\mu(i)} \neq a_{i\mu(i)}$. This is why it is enough to consider transfers between coalitions $S \subseteq M$ and $T \subseteq M'$ such that $T = \mu(S)$ for all $\mu \in \mathcal{M}^*_\gamma(M, M')$. Thus, for a given square assignment market $\gamma = (M, M', A, p, q)$, we define

$$\mathcal{S}(\gamma) = \left\{\emptyset \neq S \subseteq M \,|\, \mu(S) = \mu'(S) \text{ for all } \mu, \mu' \in \mathcal{M}^*_\gamma(M, M')\right\}. \tag{23}$$

Theorem 5.1. *Let* $\gamma = (M, M', A, p, q)$ *be a square assignment market and* $\mu \in \mathcal{M}^*_\gamma(M, M')$. *Then, the nucleolus is the unique core allocation satisfying* $\delta_{S,\mu(S)}(\eta(w_\gamma)) = \delta_{\mu(S),S}(\eta(w_\gamma))$, *for all* $S \in \mathcal{S}(\gamma)$.

Proof. Let $\gamma = (M, M', A, p, q)$ be a square assignment market, $\eta = \eta(w_\gamma)$ its nucleolus, and $S \in \mathcal{S}(\gamma)$. Notice first that, for all $x \in C(M \cup M', w_\gamma)$ and all $\mu \in \mathcal{M}^*_\gamma(M, M')$, $\delta_{S,\mu(S)}(x) = \min_{\substack{i \in S \\ k \in M' \setminus \mu(S)}} \{x_i - p_i, x_i + x_k - a_{ik}\}$ and $\delta_{\mu(S),S}(x) = \min_{\substack{j \in \mu(S) \\ k \in M \setminus S}} \{x_j - q_j, x_k + x_j - a_{kj}\}$. Thus, $\delta_{S,\mu(S)}(x) = \delta_{\mu(S),S}(x)$ is equivalent to $\max_{\substack{i \in S \\ k \in M' \setminus \mu(S)}} \{p_i - x_i, a_{ik} - x_i - x_k\} = \max_{\substack{j \in \mu(S) \\ k \in M \setminus S}} \{q_j - x_j, a_{kj} - x_j - x_k\}$. Expression (17) shows that the nucleolus satisfies the above equality. Suppose there is $z \in C(M \cup M', w_\gamma)$, $z \neq \eta$, such that $\delta_{S,\mu(S)}(z) = \delta_{\mu(S),S}(z)$, for all $S \in \mathcal{S}(\gamma)$ and all $\mu \in \mathcal{M}^*_\gamma(M, M')$. Since for all $S \notin \mathcal{S}(\gamma)$, $\delta_{S,\mu(S)}(z) = \delta_{\mu(S),S}(z) = 0$, we have that $\delta_{S,\mu(S)}(z) = \delta_{\mu(S),S}(z)$, for all $S \subseteq M$ and all $\mu \in \mathcal{M}^*_\gamma(M, M')$. Now uniqueness follows by using the same argument that leads to expression (18) in Theorem 4.1's proof. □

References

[1] Davis M, Maschler M (1965) The kernel of a cooperative game. Naval Research Logistics Quarterly, 12, 223-259.

[2] Driessen TSH (1998) A note on the inclusion of the kernel in the core of the bilateral assignment game. International Journal of Game Theory, 27, 301–303.

[3] Maschler M, Peleg B, Shapley S (1979) Geometric properties of the kernel, nucleolus and related solution concepts. Mathematics of Operations Research, 4, 303-338.

[4] Rochford SC (1984) Symmetrically pairwise-bargained allocations in an assignment market. Journal of Economic Theory 34: 262-281.

[5] Shapley LS, Shubik M (1972) The Assignment Game I: The Core. International Journal of Game Theory 1, 111–130.

[6] Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM Journal of Applied Mathematics 17, 1163–1170

[7] Solymosi T, Raghavan TES (1994) An algorithm for finding the nucleolus of assignment games. International Journal of Game Theory 23, 119–143.