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DEPARTAMENT D'ECONOMIA – CREIP
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The equity core and the core

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Abstract

In this paper we study the equity core (Selten, 1978) and compare it with the core. A payoff vector is in the equity core if no coalition can divide its value among its members proportionally to a given weight system and, in this way, give more to each member than the amount he or she receives in the payoff vector. We show that the equity core is a compact extension of the core and that, for non-negative games, the intersection of all equity cores with respect to all weights coincides with the core of the game.

Keywords: Cooperative game, equity core, equal division core, core.
JEL classification: C71

1. Introduction

Selten (1972) introduced the equal division core as an extension of the core. Selten shows experimentally that the outcome of the game will have a strong tendency to be in the equal division core. There are in the literature two main explanations for the equal division core. On the one hand, Selten (1972) used the equal division core to explain outcomes of experimental cooperative games. Selten (1987) argues that the evidence suggests that equity considerations have a strong influence on observed payoff divisions. On the other hand, Dutta and Ray (1991) follow a theoretical approach to obtain the equal division core as a set solution of a game satisfying participation constraints for coalitions if the norm of egalitarianism is used consistently for coalitions.

Selten (1972) assumes that all the players have the same weight, but in many situations this could be restrictive. The equity core of a transferable utility coalitional game was introduced by Selten (1978) as a weighted generalization of the previous notion of equal division core (Selten, 1972), by taking into account exogenous and positive weights of the players. When all players have the same weight both notions coincide.

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The equity core, with respect to a positive vector, $w \in \mathbb{R}^N_{++}$, of weights is the set of efficient payoff vectors such that no coalition can divide its value proportionally to w among its members and, in this way, give more to all its members than the amount they receive in the payoff vector.

To motivate the equity core and compare it with the equal division core, we consider the financial cooperative games introduced by Izquierdo and Rafels (2001). Let $N = \{1, \ldots, n\}$, be a group of investors such that each of them has an amount of money $w_i > 0, i = 1 \ldots, n$ to invest. Suppose a bank offers a yield that increasingly depends on the amount of money invested and assume that investors may combine their resources, $w(S) = \sum_{i \in S} w_i,\ S \subseteq N$, and invest them in the bank. The characteristic function is given by $v(S) = w(S)i(w(S))$ where $i(w(S))$ is the yield that coalition S gets by joining their resources. Next we give an example.

Example 1.1. *Let (N, v) be a three-person cooperative game, where the players are a group of investors. The amounts each agent may invest are, $c_1 = 750$, $c_2 = 250$ and $c_3 = 250$. Suppose that the yield offered by the bank is given by*

$$i(c) = \begin{cases} 0\% & \textit{if}\, c < 1000, \\ 10\% & \textit{if}\, c \geq 1000, \end{cases}$$

where c is the amount of money invested.

The characteristic function of the game is the following: $v(i) = 0$ for all $i \in N = \{1, 2, 3\}$, $v(12) = v(13) = 100$, $v(23) = 0$ and $v(N) = 125$. The equal division core is the set:

$$\begin{aligned} EDC(N, v) = \{x \in I(v) \mid\ & (x_1 \geq 50 \text{ or } x_2 \geq 50) \text{ and} \\ & (x_1 \geq 50 \text{ or } x_3 \geq 50) \text{ and } (x_2 \geq 0 \text{ or } x_3 \geq 0)\}. \end{aligned}$$

Consider the allocation $(50, 0, 75) \in EDC(N, v)$. Under the above interpretation of the game, this means that player 1 receives $6,\hat{6}\%$ of what he invests, player 2 receives 0% and player 3 receives 30%. Notice that this allocation is objectionable because player 1, which invested the greatest amount of money, receives a very low yield. Moreover, player 2, who invested the same amount than player 3, does not receive anything. Other allocations in the equal division core have the same problem. To be precise this happens either to all the allocations in the zones C and D of Figure 1. The yield that every player receives in each zones is:

$$\text{In zone C:} \begin{cases} 6,\hat{6}\% & \leq \text{yield of player } 1 \leq 10\% \\ 0\% & \leq \text{yield of player } 2 \leq 10\% \\ 10\% & \leq \text{yield of player } 3 \leq 30\% \end{cases}$$

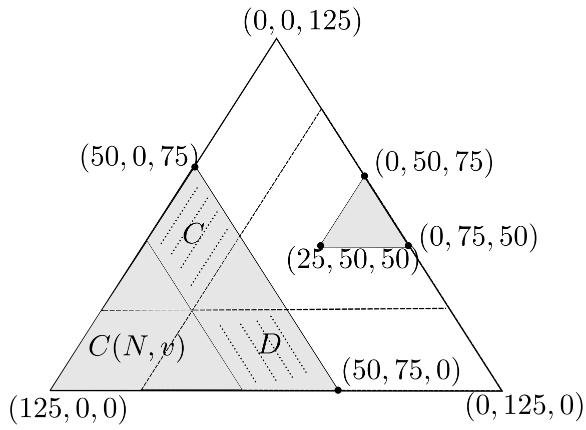

Figure 1: This figure corresponds to Example 1.1. The two shadowed zones in the triangle are the equal division core.

$$\text{In zone D: } \begin{cases} 6,\hat{6}\% & \leq \text{yield of player 1} \leq 10\% \\ 10\% & \leq \text{yield of player 2} \leq 30\% \\ 0\% & \leq \text{yield of player 3} \leq 10\% \end{cases}$$

In both zones C and D only one player, player 3 or player 2, receives a yield higher than 10%, penalizing player 1, who receives less than 10% in both zones. This could make sense if players 2 and 3 both could have profits, but it is debatable that only one of them has all the profits.

Next we compute the core and the equity core for this game considering as a weight for every player the amount he or she wishes to invest, so $w_i = c_i$ for all $i \in N$. The core of the game is $C(v) = \{x \in I(v) \mid x_2 \leq 25, x_3 \leq 25\}$ and the equity core with respect to the amount of capital invested $w = (750, 250, 250)$ is:

$$EC^w(N, v) = \{x \in I(N, v),\ x_1 \geq 75 \text{ or } (x_2 \geq 25 \text{ and } x_3 \geq 25)\ \}.$$

In Figure 2 we have represented $EC^w(N, v)$. Observe that the equity core does not include the two zones C and D of the previous Figure 1 which, as we have said before, are debatable. Whereas, if we analyze the different zones in the equity core we will see that all of them have sense, although some vectors can be considered better than others from some point of view.

Let us analyze the yield received by the players in each one of the zones in the equity core.

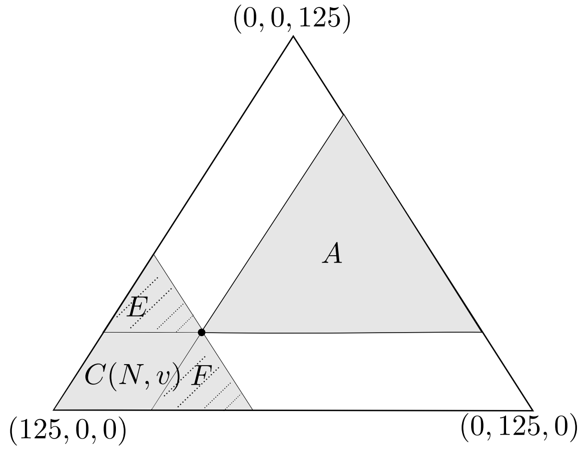

Figure 2: The shadowed zone in the triangle is the equity core of Example 1.1.

$$\text{In zone A:} \begin{cases} 0\% & \leq \text{yield of player } 1 \leq 10\% \\ 10\% & \leq \text{yield of player } 2 \leq 40\% \\ 10\% & \leq \text{yield of player } 3 \leq 40\% \end{cases}$$

In this zone both players 2 and 3 receive more than 10% and the only one who receives less than 10% is player 1.

$$\text{In zone E:} \begin{cases} 10\% & \leq \text{yield of player } 1 \leq 13,3\% \\ 0\% & \leq \text{yield of player } 2 \leq 10\% \\ 10\% & \leq \text{yield of player } 3 \leq 20\% \end{cases}$$

In this zone player 1 and 3 receive more than 10% and the only one who receives less than 10% is player 2.

$$\text{In zone F:} \begin{cases} 10\% & \leq \text{yield of player } 1 \leq 13,3\% \\ 10\% & \leq \text{yield of player } 2 \leq 20\% \\ 0\% & \leq \text{yield of player } 3 \leq 10\% \end{cases}$$

In this zone player 1 and 2 receive more than 10% and the only one who receives less than 10% is player 3.

$$\text{In the core zone:} \begin{cases} 10\% & \leq \text{ yield of player } 1 \leq 16,6\% \\ 0\% & \leq \text{yield of player } 2 \leq 10\% \\ 0\% & \leq \text{yield of player } 3 \leq 10\% \end{cases}$$

In the core zone player 1 is the only one that receives more than 10% and both players 2 and 3 receive less than 10%.

Observe that in the equity core either player 1 is one of the players that receive a yield over the 10% or players 2 and 3 together receive more than 10%. Hence, in all cases this is coherent with the capital invested by the players.

In many real situations not all the players have the same weight in the game, we think on problems where there are some type of exogenous priority, given by the law (like bankruptcy problems, merits in a public contest) or by custom (women and children first). Considering these situations, in this work we analyze the equity core and its relation with the core (Gillies, 1953), which is one of the most important solution concepts in cooperative game theory.

As for the equal division core, we show that the equity core is also a compact extension of the core and coincides with the core of a particular non-transferable utility game. Moreover they are clearly different solution sets. Next we show that for any non-negative game, the intersection of all the equity cores associated with a positive vector of weights coincides with the core. As a consequence, since in convex games the classical *marginal worth vectors* are extreme points of the core, so they are in the equity core for any vector of weights $w \in \mathbf{R}^n_{++}$. Thus, a characterization of the convexity for non-negative games is provided. Finally we give a condition that guarantees the coincidence between the equity core and the core.

The paper is organized as follows. In Section 2 we introduce some preliminaries and basic definitions. In Section 3 we study and compare the equity core with the core. Section 4 has the final remarks.

2. Preliminaries

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. By $N \subseteq \mathbb{N}$ we denote a finite set of players; in general $N = \{1, \ldots, n\}$. A *transferable utility coalitional game (a game)* is a pair (N, v) where $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function, with $v(\emptyset) = 0$, and 2^N denotes the set of all subsets (coalitions) of N. A game (N, v) is *non-negative* if $v(S) \geq 0$, for any coalition $S \subseteq N$. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. The set of all games is denoted by Γ. To avoid cumbersome notation we will write $v(ij \ldots k)$ instead of $v(\{i, j, \ldots, k\})$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. We define $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \mid x \gg 0\}$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ means $x_i \geq y_i$, for all $i \in N$, and $x \gg y$ denotes that $x_i > y_i$ for all $i \in N$. We say that $x > y$, if and only if $x \geq y$ and for some $j \in N$, $x_j > y_j$.

The *pre-imputation set* of a game (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$. A *solution* on a set Γ of games is a mapping σ which associates with any game (N, v) a subset $\sigma(N, v)$ of the set $X(N, v)$. Notice that the solution set $\sigma(N, v)$ is allowed to be empty. For a game (N, v), the set of *imputations* is given by $I(N, v) := \{x \in X(N, v) \,|\, x(i) \geq$

$v(i),$ for all $i \in N\}$. The *core* of a game (N, v) is the set of those imputations where each coalition gets at least its worth, that is $C(N, v) := \{x \in X(N, v) \mid x(S) \geq v(S)$ for all $S \subseteq N\}$. The *equal division core* (Selten, 1972) is an extension of the core containing those imputations which can not be improved upon by the equal division allocation of any subcoalition; formally $EDC(N, v) := \left\{x \in I(N, v) \mid \text{for all } \emptyset \neq S \subseteq N, \text{ there is } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|}\right\}$. A game (N, v) is *convex* (Shapley, 1971) if, for every $S, T \subseteq N$, $v(S)+v(T) \leq v(S \cup T)+v(S \cap T)$.

An ordering $\theta = (i_1, \ldots, i_n)$ of N, where $|N| = n$, is a bijection from $\{1, \ldots, n\}$ to N. We denote by $\mathcal{S}_N$ the set of all orderings of N. Given a game (N, v) and an ordering $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$, we define the *marginal worth vector* associated with θ as the vector $m^\theta(v) \in \mathbb{R}^N$ that assigns to each player his o her marginal contribution in the order θ. Formally, $m^\theta_{i_1}(v) = v(i_1)$, and $m^\theta_{i_k}(v) = v(i_1, \ldots, i_k) - v(i_1, \ldots, i_{k-1})$, for $k = 2, \ldots, n$.

3. The equity core and the core

In this section we analyze the equity core with respect to $w \in \mathbb{R}^N_{++}$, a list of positive weights which represents an exogenous asymmetry between the players of N. We show that, as in the equal division core, the equity core is also a compact extension of the core. In addition, we prove that for non-negative games, the intersection of all the equity cores with respect to all possible weights $w \in \mathbb{R}^N_{++}$ coincides with the core of the game.

Definition 3.1. *Let (N, v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. The equity core w.r.t. w is:*

$$EC^w(N, v) := \left\{x \in I(N, v) \,|\, \text{for all } \emptyset \neq S \subseteq N \text{ there is } i \in S \text{ with } x_i \geq \frac{v(S)}{w(S)} w_i\right\}.$$

The equity core is an asymmetric extension of the equal division core in which the players may have different weights. A payoff vector is in the equity core if no coalition can divide its worth proportionally to w among its members and, in this way, give more to each of them than the amount they receive in the payoff vector.

Llerena and Vilella (2013) show that the equity core can be decomposed as a finite union of polyhedrons.

Definition 3.2. *Let (N, v) be a game, $w \in \mathbb{R}^N_{++}$ a vector of weights and $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$. The proportional share worth vector w.r.t. w and θ, denoted by $x^{\theta,w} \in \mathbb{R}^N$, is:*

$$x^{\theta,w}_{i_k} := \max_{S \in P_{i_k}} \left\{\frac{v(S)}{w(S)}\right\} w_{i_k}, \text{ for } k = 1, \ldots, n,$$

where $P_{i_1} := \{S \subseteq N \mid i_1 \in S\}$ and $P_{i_k} := \{S \subseteq N \mid i_1, \ldots, i_{k-1} \notin S, \; i_k \in S\}$, for $k = 2, \ldots, n$.

To simplify notation, from now on we write x^θ instead of $x^{\theta,w}$.

Definition 3.3. *Let (N, v) be a game, $w \in \mathbb{R}^N_{++}$ a vector of weights and $\theta = (i_1, \ldots, i_n) \in \mathcal{S}_N$. The polyhedron generated by x^θ, denoted by Δ^{x^θ} is:*

$$\Delta^{x^\theta} := convex\,\{x^\theta + \delta_{x^\theta}\, e_i,\ for\ all\, i \in N\},$$

where $\delta_{x^\theta} = v(N) - x^\theta(N)$ and e_i is the i-th canonical vector of $\mathbb{R}^N$ for any $i \in N$.

Definition 3.4. *Let (N, v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. The set of minimal proportional share worth vectors is:*

$$\mathcal{M}^w(v) := \{x^\theta \,|\, \theta \in \mathcal{S}_N,\ \delta_{x^\theta} \geq 0\, and\ there\ is\ no\ \ \theta'\ such\ that\ x^{\theta'} \leq x^\theta\}.$$

Theorem 3.5. *(Llerena and Vilella, 2013) Let (N, v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then,*

$$EC^w(N, v) = \bigcup_{x^\theta \in \mathcal{M}^w(v)} \Delta^{x^\theta}.$$

Selten (1972) proved that the equal division core is an extension of the core. It is easy to see that the equity core is also a compact extension of the core, but there are relevant differences between them (for more details see Vilella (2004)).

Proposition 3.6. *Let (N, v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then, the equity core with respect to w is a compact extension of the core, i.e.*

$$C(N, v) \subseteq EC^w(N, v).$$

Proof: Suppose $x \in C(N, v)$ and $x \notin EC^w(N, v)$. Then, there exists a coalition $\emptyset \neq S \in 2^N$ such that $x_i < \frac{v(S)}{w(S)} w_i$ for all $i \in S$, then $x(S) < v(S)$ which is a contradiction to the assumption that x is in the core. By definition, the $EC^w(N, v)$ is bounded and since it is a finite intersection of a finite union of closed sets (see Theorem 3.5), it is closed too, so it is a compact set. □

Proposition 3.7 next shows that, for any game (N, v) the equity core coincides with the core of a certain non-transferable utility game (NTU game) (N, V) (for formal definitions of an NTU game and the core, see Kannai (1992)). This connection was pointed out by Bhattacharya (2004) for the particular case of the EDC.

Proposition 3.7. *Let (N, v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Let (N, V) be an NTU game where $V(N) = \{x \in \mathbb{R}^N \mid x \leq y$ for some $y \in X(N, v)\}$, $V(S) = \{x \in \mathbb{R}^S \mid x_i \leq \frac{v(S)}{w(S)} w_i$ for all $i \in S\}$ and $V(\emptyset) = \emptyset$. Then,*

$$EC^w(N, v) = C(N, V).$$

Proof: Let $x \in EC^w(N, v)$. To prove that $x \in C(N, V)$, first we show that $x \in V(N)$. Let $y = x$, since $x(N) = v(N)$ then $y(N) = v(N)$. Therefore, $y \in X(N, v)$ and $x \leq y$. Thus,

$x \in V(N)$. Next, assume there exists a coalition $S \subseteq N$ and $y \in \mathbb{R}^S$ such that $y_i > x_i$ for all $i \in S$ and $y \in V(S)$ which implies $y_i \leq \frac{v(S)}{w(S)} w_i$ for all $i \in S$. As a consequence, $x_i < y_i \leq \frac{v(S)}{w(S)} w_i$ for all $i \in S$, which is a contradiction to $x \in EC^w(N, v)$. Therefore, $x \in C(N, V)$.

To prove the reverse inclusion, let $x \in C(N, V)$. First, we prove that $x(N) = v(N)$. Assume $x(N) \neq v(N)$. Since $x \in V(N)$ then, $\delta = V(N) - x(N) \geq 0$. Let $y = (x_i + \frac{\delta}{n})_{i \in N}$, then $y(N) = v(N)$ and $y \in V(N)$. Therefore, for $S = N$, $y \in \mathbb{R}^n$ and $y_i \geq x_i$. If $y_i = x_i$ for all $i \in N$ then $x(N) = v(N)$. Otherwise $y_i > x_i$ for all $i \in N$, which is a contradiction to $x \in C(N, V)$. Therefore, $x(N) = v(N)$. Next, assume there exists a coalition $S \subseteq N$ such that $x_i < \frac{v(S)}{w(S)} w_i$ for all $i \in S$. Let $y = (\frac{v(S)}{w(S)} w_i)_{i \in S}$. Then $y \in \mathbb{R}^S$, $y \in V(S)$ and $x_i < y_i$ for all $i \in S$, which contradicts $x \in C(N, V)$. Therefore, there is no coalition $S \subseteq N$ such that $x_i < \frac{v(S)}{w(S)} w_i$ for all $i \in S$ and thus $x \in EC^w(N, v)$. □

Theorem 3.8 states that for any non-negative game, the intersection of all the equity cores associated to a positive vector of weights coincides with the core.

Theorem 3.8. *Let (N, v) be a non-negative game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then,*

$$\bigcap_{w \in \mathbb{R}^N_{++}} EC^w(N, v) = C(N, v).$$

Proof: Since the equity core is an extension of the core, the inclusion $C(N, v) \subseteq \bigcap_{w \in \mathbb{R}^N_{++}} EC^w(N, v)$ follows straightforwardly.

To show the reverse inclusion, let $x \in \bigcap_{w \in \mathbb{R}^N_{++}} EC^w(N, v) \setminus C(N, v)$. Hence, $x_i \geq v(i) \geq 0$, for all $i \in N$, and there exists a coalition $S \neq \emptyset, N$, such that $x(S) < v(S)$. Now define the vector of weights $w' \in \mathbb{R}^N_{++}$ as follows:

$$w'_i := \begin{cases} x_i + \dfrac{v(S) - x(S)}{|S|} & \text{if } i \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Notice that $w'(S) = v(S)$. Since $x \in \bigcap_{w \in \mathbb{R}^N_{++}} EC^w(N, v)$, in particular $x \in EC^{w'}(N, v)$, and so $x_i \geq \frac{v(S)}{w'(S)} w'_i = w'_i > x_i$, which is a contradiction. Thus, $x \in C(N, v)$. □

The next example proves that the non-negativity condition is necessary for the above theorem to hold.

Example 3.9. *Let (N, v) be a three-person game, where $N = \{1, 2, 3\}$ and $v(i) = -1$ for all $i \in N$, $v(S) = 0$, for all $S \subset N$ with $|S| = 2$, and $v(N) = 2$. The core of this game is $C(N, v) = \{(x_1, x_2, x_3) \in I(N, v) \mid x_1 \leq 2, x_2 \leq 2, x_3 \leq 2\}$. It is easy to see that for any vector of weights $w \in \mathbb{R}^N_{++}$ all the equity cores coincide: $EC^w(N, v) = \{(x_1, x_2, x_3) \in I(N, v) \mid (x_1 \geq 0 \text{ or } x_2 \geq 0)$ and $(x_1 \geq 0 \text{ or } x_3 \geq 0)$ and $(x_2 \geq 0 \text{ or } x_3 \geq 0)\}$. Therefore, $\bigcap_{w \in \mathbb{R}^N_{++}} EC^w(N, v) = EDC(N, v) \neq C(N, v)$.*

As a direct consequence of Theorem 3.8 we have a characterization of the convexity of non-negative games in terms of the marginal worth vectors and the equity core.

Corollary 3.10. *Let (N, v) be a non-negative game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then, the following statements are equivalent:*

1. *(N, v) is convex.*
2. *For all $\theta \in \mathcal{S}_N$, $m^\theta(v) \in EC^w(N, v)$.*

Proof. The first implication is an straight consequence of Proposition 3.6 and the fact that, for convex games all marginal worth vectors are in the core.

For the second implication, assuming that for all $\theta \in \mathcal{S}_N$ and for all $w \in \mathbf{R}^n_{++}$, $m^\theta(v) \in EC^w(N, v)$, we have $m^\theta(v) \in \bigcap_{w \in \mathbf{R}^n_{++}} EC^w(N, v)$. Since the game is nonnegative, by Theorem 3.8 we have $\bigcap_{w \in \mathbf{R}^n_{++}} EC^w(N, v) = C(N, v)$. Thus, $m^\theta(v) \in C(N, v)$ for all $\theta \in \mathcal{S}_N$. Ichiishi (1981) proves that if all the marginal worth vectors are in the core the game is convex. Then, the game (N, v) is convex. $\square$

Equality between the equity core and the core rarely occurs. As a consequence of Theorem 3.5, if for all the proportional share worth vectors each coalition gets at least its worth, we can guarantee their equality.

Corollary 3.11. *Let (N, v) be a game and $w \in \mathbb{R}^N_{++}$ a vector of weights. Then, the following statements are equivalent:*

1. *$C(N, v) = EC^w(N, v)$.*
2. *For all $\theta \in \mathcal{S}_N$ such that $\delta_{x^\theta} \geq 0$ and for all $S \subset N$, $x^\theta(S) \geq v(S)$.*

Proof: Suppose $C(N, v) = EC^w(N, v)$. If there is no $\theta \in \mathcal{S}_N$ with $\delta_{x^\theta} \geq 0$, then it is proved. Otherwise, let $\theta \in \mathcal{S}_N$ such that $\delta_{x^\theta} \geq 0$. Then, $x^\theta + \delta_{x^\theta} e_i \in C(N, v)$, for all $i \in N$. Let $S \subset N$ and take $j \in N \setminus S$. Then, $x^\theta(S) = x^\theta(S) + \delta_{x^\theta} e_j(S) \geq v(S)$ for all $S \subset N$.

Conversely, if there is no $\theta \in \mathcal{S}_N$ with $\delta_{x^\theta} \geq 0$ then $EC^w(N, v) = \emptyset$. Thus, since the core is included in the equity core, $C(N, v) = EC^w(N, v)$. Otherwise, take $\theta \in \mathcal{S}_N, \delta_{x^\theta} \geq 0$. On one hand by Theorem 3.5 we have that

$$EC^w(N, v) = \bigcup_{x^\theta \in \mathcal{M}^w(v)} \Delta^{x^\theta} = \bigcup_{x^\theta \in \mathcal{M}^w(v)} convex\{x^\theta + \delta_{x^\theta} e_i, i \in N\}. \tag{1}$$

On the other hand, by assumption we have that

$$\begin{aligned} &\text{if } i \in S,\ x^\theta(S) + \delta_{x^\theta} e_i(S) = x^\theta(S) + \delta_{x^\theta} \geq x^\theta(S) \geq v(S), \\ &\text{if } i \notin S,\ x^\theta(S) + \delta_{x^\theta} e_i(S) = x^\theta(S) \geq v(S). \end{aligned} \tag{2}$$

Thus, in both cases, $x^\theta + \delta_{x^\theta} e_i \in C(N, v)$. Since the core is a convex set, the convex hull of these points will be in the core too, i.e. $\Delta^{x^\theta} \subseteq C(N, v)$. From expression (1) we have, $EC^w(N, v) \subseteq C(N, v)$ and since $C(N, v) \subseteq EC^w(N, v)$, we have $C(N, v) = EC^w(N, v)$. $\square$

4. Final remarks

Selten considers the equity core in a more general framework where there is a coalition structure $(B_1, \ldots, B_k)$ of the players and the set of feasible coalitions can be restricted to $\mathcal{P} \subseteq 2^N$. In this work we consider the particular case where the coalition structure is the grand coalition N and all coalitions are feasible. This is so in order to be able to compare, graphically and analytically, the relative position between the classical core and the equity core and to simplify notation. Moreover, most of the previous results like Proposition 3.6 and Theorem 3.8 can be extended to this more general framework (for more details see Vilella, 2004).

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